К 105-летию Финансового университета при Правительстве РФ

Д.В. Берзин, С.А. Зададаев

РЕШЕНИЕ ЗАДАЧ В ТАБЛИЧНОМ ПРОЦЕССОРЕ

Учебник для вузов

Учебник представляет собой 14 компьютерных практикумов по изучению и применению вычислительных возможностей табличного процессора MS Excel в решении базовых задач линейной алгебры и математического анализа и соответствует программам дисциплин «Цифровая математика на языке R и Excel» и «Компьютерный практикум» бакалавриата (направления подготовки 38.03.01 «Экономика», 38.03.02 «Менеджмент»).

В учебнике последовательно излагаются общие характеристики табличного процессора MS Excel с углублением по мере изучения основных положений математического анализа и линейной алгебры. Отдельное внимание уделяется построению графиков функций. Показаны возможности решения задач линейного программирования. Приведены примеры решения финансово-экономических и управленческих задач. В каждом разделе представлены задания для самостоятельной работы.

Соответствует требованиям Федерального государственного образовательного стандарта высшего образования последнего поколения.

Учебник будет полезен студентам программ бакалавриата, изучающим линейную алгебру и математический анализ, которые стремятся освоить инструментальные средства табличного процессора MS Excel и их применение при решении финансово-экономических и управленческих задач с помощью инновационных математических методов и технологий. Учебник может быть интересен магистрантам, аспирантам, преподавателям и научным сотрудникам.

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Издательство Прометей

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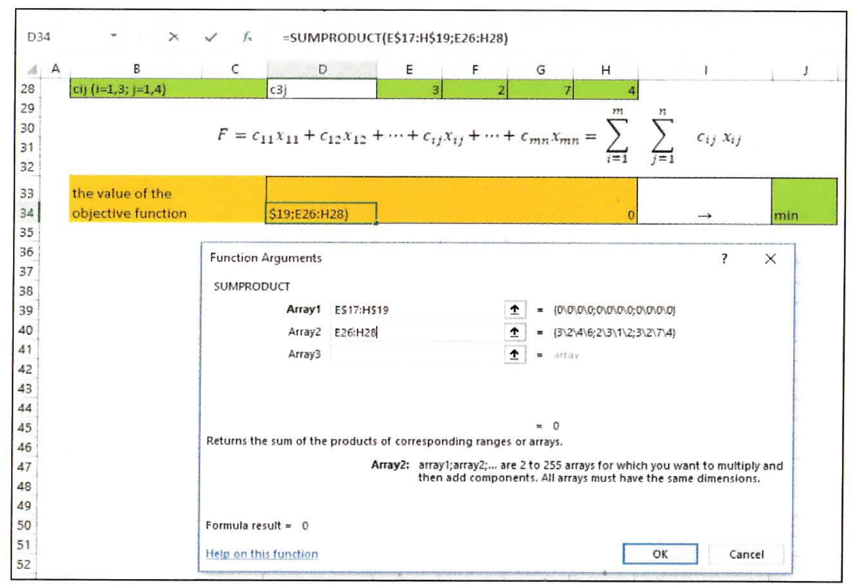

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Д.В. Берзин, С.А. Зададаев

Решение задач в табличном процессоре

Учебник для вузов

Москва
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В учебнике рассматриваются вопросы использования табличного процессора MS Excel при решении ряда задач математического анализа, линейной алгебры, линейного программирования, а также финансово-экономических задач. Учебник предназначен прежде всего для студентов первого года обучения программы бакалавриата (направления 38.03.01 «Экономика», 38.03.02 «Менеджмент»), изучающих дисциплины «Цифровая математика на языке R и Excel» и «Компьютерный практикум».

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«FINANCIAL UNIVERSITY
UNDER THE GOVERNMENT OF THE RUSSIAN FEDERATION»
Department of Mathematics

D. Berzin, S. Zadadaev

Solving problems in spreadsheets

Textbook

ИЗДАТЕЛЬСТВО ПРОМЕТЕЙ

Moscow
2024

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Structurally, this textbook consists of 14 computer workshops on computational capabilities of the MS Excel in solving problems of Calculus, Linear Algebra, and Linear Programming. It corresponds to the syllabuses of the disciplines "Computer Workshop" and "Digital Mathematics in MS Excel and R" taught at the Financial University under the Government of the Russian Federation in the first year of general economic and management specialties (bachelor program of studies, directions 38.03.01. «Economics», 38.03.02 «Management»).

The textbook will be helpful for students who study Calculus, Linear Algebra, Linear Programming, and Economics and want to know the most modern computing technologies.

Also, it will be valuable for those who want to learn how to work in MS Excel and use the tabular processor in daily business life.

The textbook can be interesting for graduate students, postgraduate students, researchers, teachers, and professors.

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Содержание

Preface

The textbook “Solving problems in speradsheets” consists of 14 chapters (computer workshops) on the study and application of the computing capabilities of the MS Excel spreadsheet processor in solving basic problems of Linear Algebra, Calculus, and Linear Programming and corresponds to the programs of the disciplines “Digital Mathematics in MS Excel and R” and “Computer Workshop” taught at the Financial University under the Government Russian Federation in the first year of undergraduate programs.

The textbook consistently outlines the general characteristics of the MS Excel spreadsheet processor with in-depth study of the basic principles of Calculus and Linear Algebra. Special attention is paid to constructing function graphs. Linear Programming and financial problems are also considered. Examples of solving economic and managerial exercises are given. Each section contains tasks for independent work.

The textbook meets the requirements of the Federal State Educational Standard of Higher Education of the latest generation.

The textbook will be useful to all undergraduate students studying Linear Algebra, Calculus, and Linear Programming who seek to master the tools of the MS Excel spreadsheet processor and their use in solving financial, economic and management problems using innovative mathematical methods and technologies. The textbook may be of interest to undergraduates, graduate students, postgraduates, teachers, professors and researchers.

Competences of the discipline "Digital Mathematics in MS Excel and R"

In the modern paradigm of Russian university education, it is customary to indicate the professional competencies developed by a particular discipline. Following this tradition, we will briefly outline what exactly we will fight for when studying the material of this textbook. The discipline provides necessary tools to form competences listed below.

	Competence	Learning outcomes (skills and knowledge) and indicators that show competence development
1.	The ability to use application software to solve professional problems.	**Know** the basic methods of obtaining, presenting, storing, and processing data using R software environment and MS Excel; **Be able to** use basic methods of obtaining, presenting, storing, and processing data by means of R and MS Excel; **Possess** the skills in solving standard problems of Calculus and Linear Algebra using the tools of R and MS Excel.
2.	The ability to apply mathematical methods to solve standard theoretical and applied problems, to interpret the obtained mathematical results.	**Know** the computational methods of the main problems of mathematical analysis and linear algebra; **Be able to** use computer technology in the implementation of mathematical methods and models for the description and analysis of applied problems; **Possess** the skills in computational work in R and communication with MS Excel.
3.	The ability to visualize analytical and reporting materials based on the results of the work performed.	**Know** the basic tools of visualizing quantitative data in R; **Be able to** use computer technologies for data presentation and graphical visualization of the results of applying mathematical methods and models to describe and analyze various applied problems; **Possess** the skills to work in RStudio in terms of visualizing quantitative data.

1. Introduction to MS Excel

General description of the MS Excel spreadsheet software

Microsoft Excel is a spreadsheet editor developed by Microsoft for Windows, macOS, Android, iOS and iPadOS. It features calculation or computation capabilities, graphing tools, pivot tables, and a macro programming language called Visual Basic for Applications (VBA). MS Excel forms part of the Microsoft 365 suite of software.

The MS Excel spreadsheet software is designed to solve multifunctional tasks of processing various types of information using a multitude of tools and built-in functions. The processing is done in an electronic spreadsheet of large size. The cells of the spreadsheet contain the original data of the applied task, as well as the formulaic dependencies that ensure the calculation of the results. The solution results are generated in the formula entry location. Optionally, they can be illustrated. When the original data is changed, the results are automatically recalculated, and the appearance of the constructed graphs and diagrams changes accordingly.

An approximate view of a spreadsheet window with a description of its elements is shown in Fig. 1.1

Depending on the version of the processor, a *workbook* defaults to one or three *worksheets*. Each worksheet of a workbook is a set of *cells* formed by 16384 columns (2^{14}) and 1048576 rows (2^{20}). Columns are named with Latin letters (from A to XFD) and rows are numbered. This allows cells to be addressed. For example, the top left cell of the sheet has the address A1, and the bottom right cell has the address XFD1048576.

One of the cells of a worksheet is active. Information entered from the keyboard or after performing an insertion op-

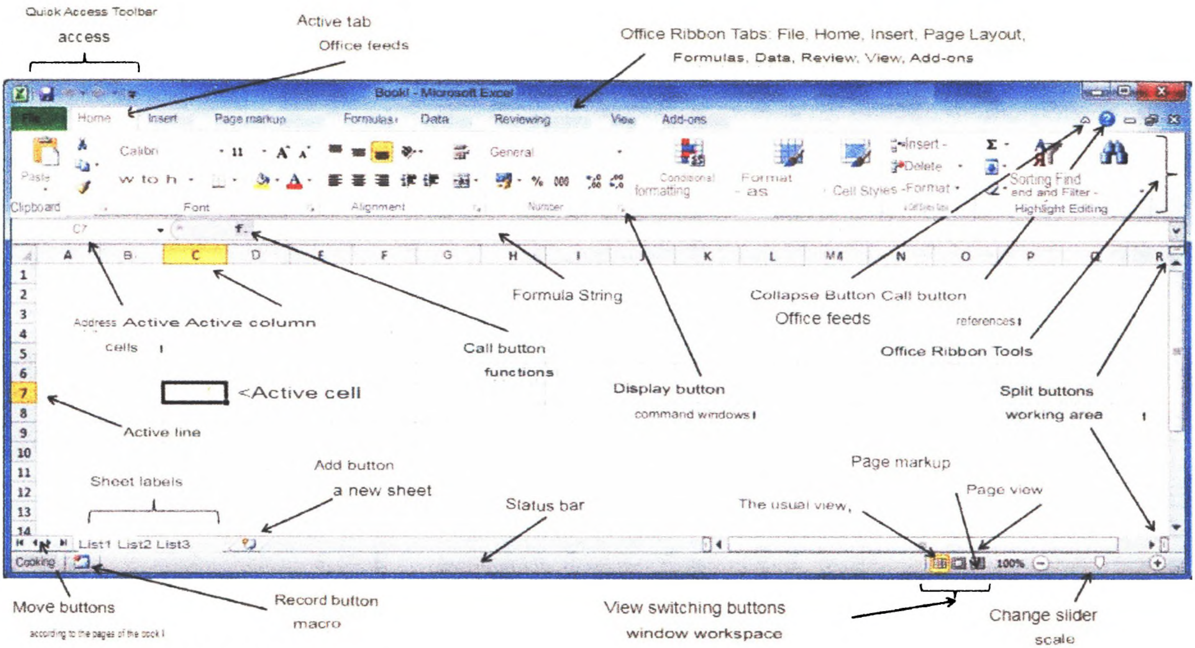


Fig. 1.1. Elements of MS Excel window

eration is recorded in this cell. The left mouse button or cursor movement keys are used to select the active cell. Quick navigation to any cell of the workbook sheet can be done using the following commands:

Press the [F5] function key → Enter the cell address in the Reference field → OK

The contents of the active cell are displayed in the formula bar. This allows you to view the information entered in the cell (especially when using formulas), and if necessary, make corrections to it.

Data can be entered not only in the active cell, but also in cell arrays — *ranges*. *Adjacent* and *non-adjacent* ranges (located on the same worksheet) and *three-dimensional* ranges (occupying the same position on several sheets) are distinguished.

In a worksheet, cell ranges are selected by moving the mouse while holding down the left button (or holding down the [Shift] key and pressing the cursor movement keys). To select non-adjacent ranges, the [Ctrl] key is additionally held down. To select a three-dimensional range, multiple sheets of the workbook are selected by clicking on their tabs while holding down the [Ctrl] key.

When recording range addresses, the address of the top-left cell and the address of the bottom-right cell are specified, separating them with a colon. Non-adjacent cell groups are separated by a semicolon (see Fig. 1.2).

Entering information into the active cell is completed by pressing the [Enter] key (or cursor movement keys). To input data into all cells of an adjacent range (as well as a portion of a non-adjacent range with the active cell), the [Ctrl+Shift+Enter] key combination is used to fix the input simultaneously.

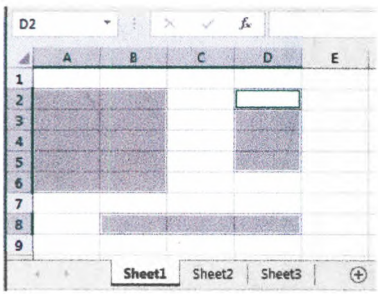


Fig. 1.2. Non-adjacent range of cells A2:B6; B8:D8; D2:D6 (cell D2 is active)

The maximum amount of data that can be entered into each cell of the workbook is $32767=2^{15}-1$ bytes (characters). However, not all characters may be displayed in the cell due to limited visible column width and the presence of data in the cell to the right. This is not necessary, as the principle of structured data entry involves placing only related, preferably indivisible data belonging to two categories — *value* or *formula* — into separate cells (Fig. 1.3).

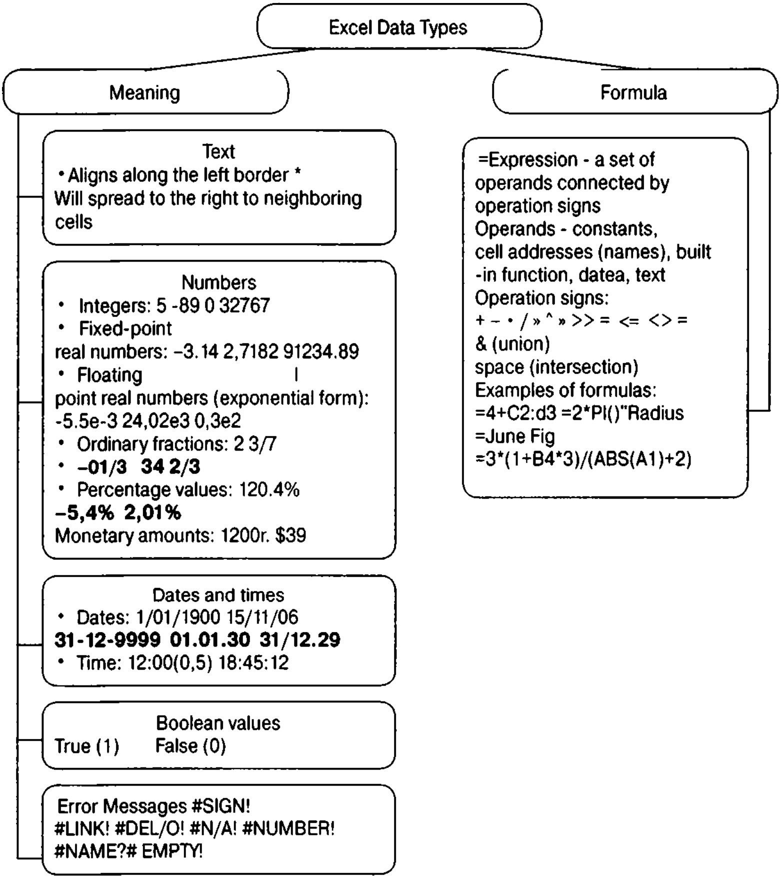


Fig. 1.3. MS Excel Data Types

The sign = (equal) entered at the beginning is an indication of entering a formula, followed by an *expression* — a set of *operands* linked together by *mathematical operators*. The order of calculations in the formula is determined by its mathematical notation (from left to right), as well as taking into account the priorities of the operations specified in it. If necessary, to change the sequence of data processing in the formula, parentheses are used.

After entering a formula in a cell, its result is immediately displayed — as a *number*, *date*, *text*, *logical value*, or *error message*. The formula itself will only be visible in the formula bar after selecting the corresponding cell.

Text, entered in a cell, is aligned to its left border, while correctly interpreted data of other types are aligned to the right border. This simple rule helps to avoid errors when inputting data (numbers, formulas, dates and times, logical values) that may be identified as text due to formatting inaccuracies.

In formulas, cell addresses can be specified as relative or absolute. Earlier in the text, relative cell addresses and their ranges were specified.

When copying formulas, relative addresses are changed accordingly, while absolute addresses remain unchanged.

A characteristic feature of an absolute address is the use of the $ symbol before the names of columns and rows. For example, $A$1 and $XFD$1048576 — the first (top left) and last (bottom right) cells of an electronic spreadsheet have absolute addresses.

Absolute addresses can be typed directly from the keyboard or converted from relative addresses after they are selected and the [F4] function key is pressed. When converting, you should be careful: pressing [F4] again can make the address relative or mixed again, for example, when only the column or only the row is fixed: $D4, G$5:F$8.

The use of cell names and their ranges also provides absolute addressing in formulas. Names must start with a letter or underscore, not contain spaces or special characters (except for numbers and periods), and be unique within the workbook.

One way to assign names to cells is:

Select the cell or range → Open the menu by right-clicking → Select the "Name a Range" option → Enter a name for the cell or range → OK

When referencing cells from other sheets or workbooks (files), their names are added to the address in expressions. For example:

[Book2]Sheet4!G5:L10 [Report]Quarter_1!$A$5:$Z$129

If sheet or workbook names contain spaces, the names should be enclosed in apostrophes:

'[Book 1]Sheet3'!$E$8' [Quarterly Report]Branch 3'!$D$2:$K$120

To avoid errors when creating expressions with names, the names are not entered using the keyboard, but rather selected in the electronic workbook from the corresponding cells or ranges, which significantly simplifies the process. The names themselves appear in the expressions.

Formatting options for electronic spreadsheets

Formatting operations can be presented schematically.

• Changing the width of a column by dragging the border between the column headings (Fig. 1.4).

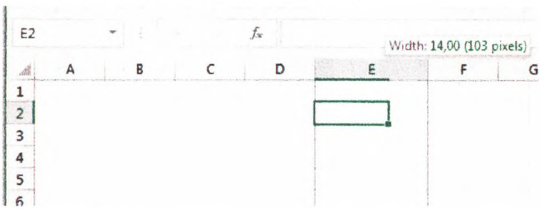


Fig. 1.4. Make column E wider.

To align the width of multiple columns, they should be selected and the border of one of them should be changed.

• Changing the height of rows is done in a similar way (Fig. 1.5).

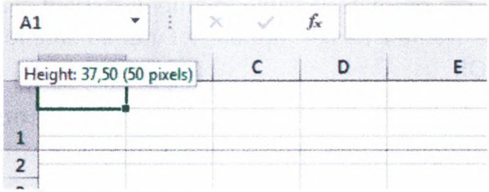


Fig. 1.5. Changing height of a cell

• To align information in selected cells, you can use the alignment buttons in the *Alignment* group on the *Home* tab or by opening the *Format Cells* window, for example, from the right-click menu (Fig. 1.6).

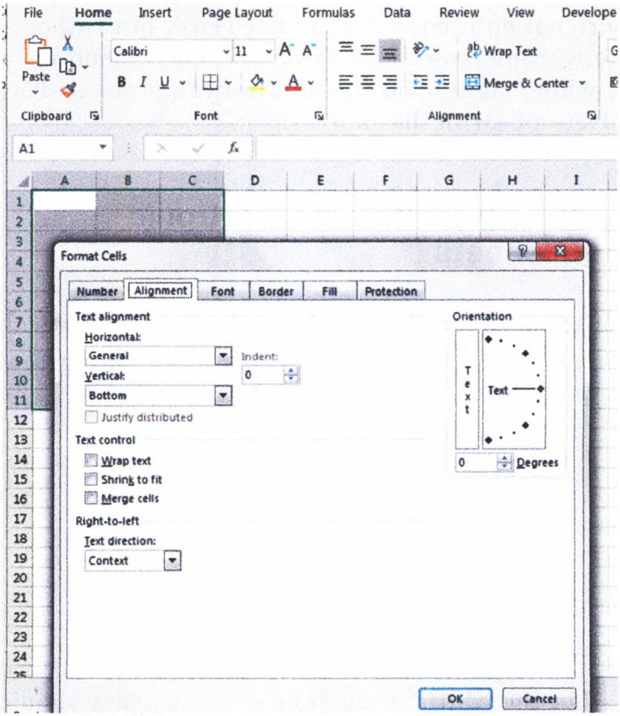


Fig. 1.6. Format Cells window

• To merge cells, you can use both the *Format Cells* window (Fig. 1.6) and the Merge & Center button in the *Alignment* group on the *Home* tab (Fig. 1.7).

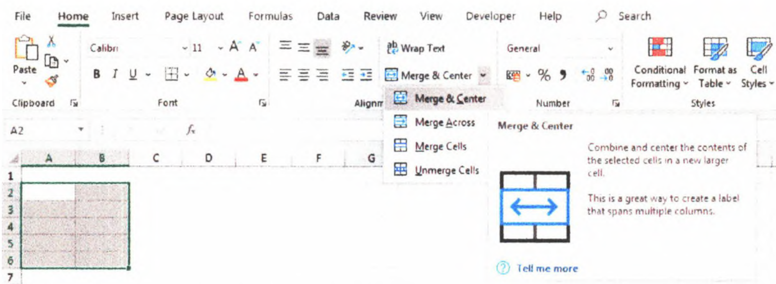


Fig. 1.7. Merging cells using the toolbar button

• Similarly, through the *Format Cells* window, you can change the orientation of text in cells, directing it at any angle (Fig. 1.8).

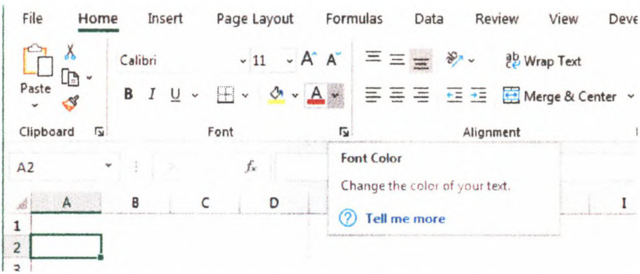


Fig. 1.8. Tool options for changing font and cell fill parameters

• Changing the font, size, and color of characters, as well as the fill and border of cells, is done using the buttons in the Font and Alignment groups on the *Home* tab (Fig. 1.8) and the options in the *Format Cells* window (Fig. 1.9, see page 16).

There are several other formatting options available in MS Excel spreadsheets, including auto-formatting, creating custom number formats, conditional formatting, setting formatting styles, inserting comments, protecting cells from changes, using sparklines, and more. These options are de-

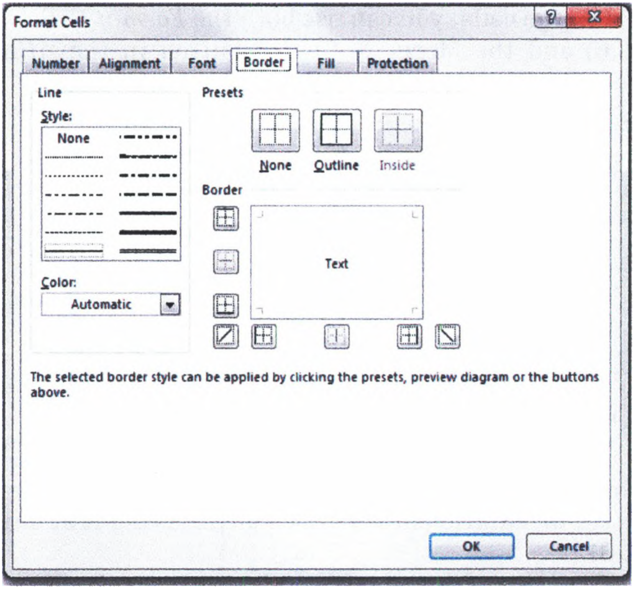


Fig. 1.9. Options for setting cell borders in an electronic spreadsheet

scribed in recommended literature, and some of them will be covered in subsequent chapter of the textbook.

Features of working with large tables

Copying (tabulating) data (formulas, sequences, constants) down a column of a large table:

Select the cell (two cells to propagate the sequence)

→ Click and drag the bottom right corner of the selected area →

Double-click the black plus sign with the left mouse button

The data in the column will be automatically tabulated to the end of the table (as long as there is some information in the left column)

We should also note some keyboard shortcuts that are useful when working with tables:

- [Ctrl+Shift+↓] — select a column of the table from the current cell in the direction of the arrow (down).
- [Ctrl+Shift+→] — select a row of the table from the current cell in the direction of the arrow (to the right).
- [Ctrl+Shift+End] — select part of the table up to the last used cell (bottom right).
- [Ctrl+C] — copy the selected cells to the Clipboard.
- [Ctrl+V] — paste data from the Clipboard (into the active cell with propagation down and to the right).
- [Ctrl+Shift+F] — change font and cell parameters in the selected range using the Format Cells window.
- [Ctrl+Shift+:] — insert the current time.
- [Ctrl+Shift+;] — insert the current date.
- [Ctrl+Shift+&] — insert external borders into the selected cell range.

Task 1.1.

The history of the Financial University dates back to March 2, 1919, when classes began at the MFEI — the first financial and economic institute in our country.

Create a MS Excel table containing information on which day of the week the anniversary of the university was (or will be) celebrated, starting from 1919 to 2019.

Highlight cells with anniversary dates: 10 years, 25 years, 50 years, 75 years, 100 years.

On which day of the week was the anniversary of the university celebrated in 1975 and 2000?

Steps of the solution.

1. Open a new MS Excel workbook.
2. Enter the text “Financial University under the Government of the Russian Federation” into cell A1.
3. Enter the text “Year of existence” into cell A3.
4. Enter the text “Date” into cell B3.
5. Enter the text “Ordinal day of the week” into cell C3.
6. Enter the text “Day of the week” into cell D3.
7. Enter the number 0 into cell A4, and the number 1 into cell A5.

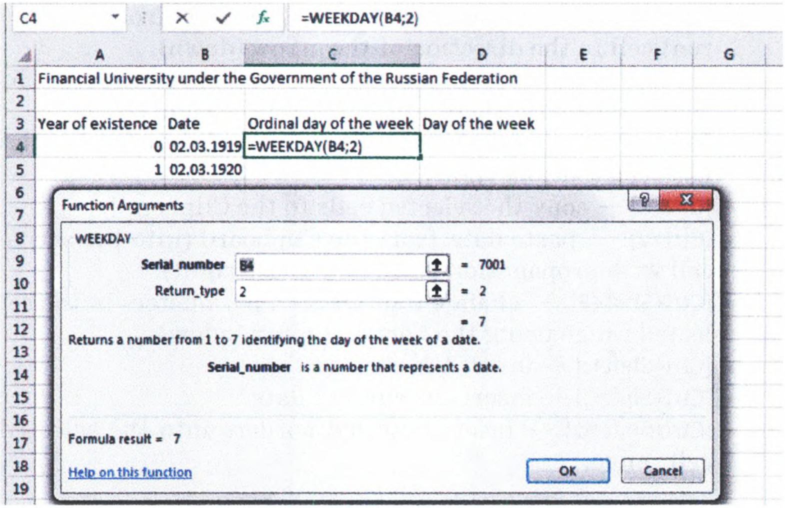


Fig. 1.10. Parameters window for the WEEKDAY function (day of the week by date)

8. Enter the date 02.03.1919 into cell B4, and the date 02.03.1920 into cell B5.
9. Enter the formula =WEEKDAY(B4;2) into cell C4. To bring up the window (Fig. 1.10), click the fx button located to the left of the formula bar.
10. Enter the names of the days of the week — Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday — in cells G4:G10 respectively.
11. Enter the numbers from 1 to 7 in cells F4:F10 respectively.
12. Select the range F4:G10
→ *Formulas tab* → *Defined Names* → *Define Name* → Enter the name *Weekdays* → *OK* (Fig. 1.12).
13. Enter the formula in cell D4:
=LOOKUP(C4,Weekdays) (Fig. 1.13).
14. Select cells A4:A5. Holding down the left mouse button, drag the bottom right corner until the cursor becomes a small black plus sign. Tabulate the data up to the number 100 (cells A104); deselect by clicking on an empty cell with the left mouse button.

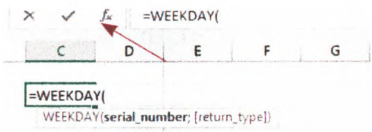


Fig. 1.11. Calling a built-in function

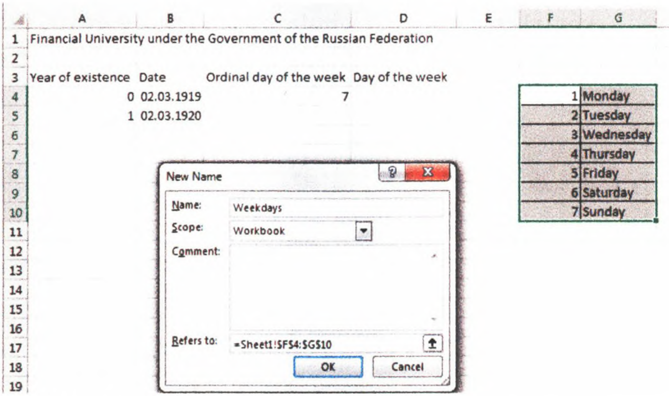


Fig. 1.12. Naming a range of cells

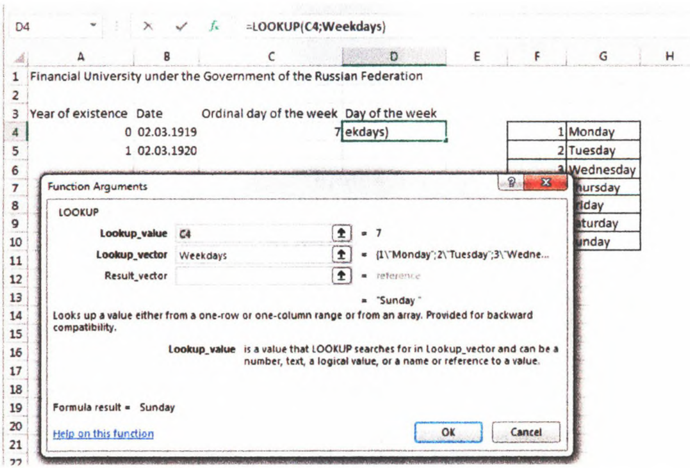
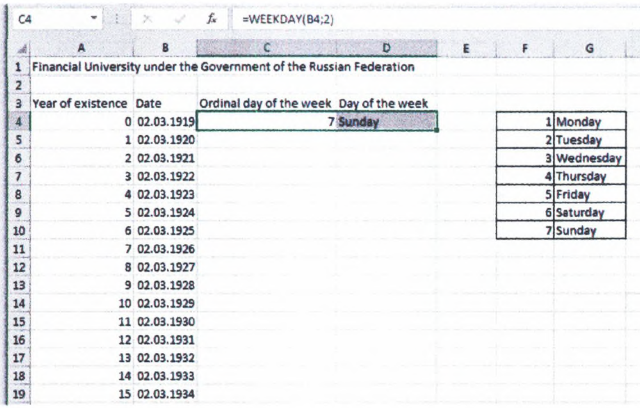


Fig. 1.13. Entering a formula with the built-in LOOKUP function

15. Select cells B4:B5. Hover the mouse cursor over the bottom right corner of the selected area until it becomes a black plus sign, then double-click the left mouse button. The data in the column will automatically be tabulated up to cell B104 (as long as there is data in the left column).
16. Select the range C4:D4. Similarly, by dragging the lower right corner, tabulate it to the end of the table (Fig. 14). The calculation of the table will be comb pleted.



C4 =WEEKDAY(B4;2)

	A	B	C	D	E	F	G
1	Financial University under the Government of the Russian Federation						
2							
3	Year of existence	Date	Ordinal day of the week	Day of the week			
4	0	02.03.1919	7	Sunday		1	Monday
5	1	02.03.1920				2	Tuesday
6	2	02.03.1921				3	Wednesday
7	3	02.03.1922				4	Thursday
8	4	02.03.1923				5	Friday
9	5	02.03.1924				6	Saturday
10	6	02.03.1925				7	Sunday
11	7	02.03.1926					
12	8	02.03.1927					
13	9	02.03.1928					
14	10	02.03.1929					
15	11	02.03.1930					
16	12	02.03.1931					
17	13	02.03.1932					
18	14	02.03.1933					
19	15	02.03.1934					

Fig. 1.14. Preparation for tabulating formulas in columns C and D

17. Select cells A1:D1 and merge them with word wrap. Change the font parameters.
18. Select cells A3:D3 and change their format, setting values to center and word wrap.
19. If necessary, adjust the width of the table columns (A:D).

    To select the entire table and set the borders surrounding the cells, do the following:

Place the cursor in cell A3 →
Select the table header with the key combination [Ctrl+Shift+→] →
Select the entire table with the key combination [Ctrl+Shift+↓] →
Set cell borders: *All borders*

20. Mark the weekdays of the anniversary celebrations in 1975 and 2000.
21. Save the file under the required name.

Exercises for independent work

1. Basing on the information provided in the chapter, construct a table that calculates the parameters of a worksheet — the number of cells and the maximum number of characters that can be placed on a worksheet (in memory units). In calculations, take the value of characters in a cell as 2^{15}.

2. Perform similar calculations for a workbook. In calculations, take the number of workbook pages as 2^{16}.

3. Evaluate what a computer RAM should be if MS Excel workbook is filled to capacity.

2. MS Excel as calculator

2.1 MS Excel expressions and operations

MS Excel is a universal computation engine that performs operations on data and uses the principle of software management. Expressions are used to write a computational program.

An expression is an indication that an action is being performed. Expressions in MS Excel are used to write formulas and filter conditions. An expression includes at least one element, such as an operator, literal, address reference, identifier, or function. Literals, identifiers, references, and functions connected by operators are called *operands*.

An operator in an expression specifies what action to perform on operands. There are arithmetic operators, comparison operators, filtration operators, concatenation operators, identification operators and address operators. The list of operators and their purpose is given in the following table. Expressions are used to write a calculation program.

Table 2.1

MS Excel operators

Operator	Perform action	Example
	Arithmetic operations	
+ (plus)	Arithmetic addition	12+24 is 36
- (minus)	Arithmetic subtraction, Unary minus (change sign of the number)	36-12 is 24
* (asterisk)	Arithmetic multiplication	13*3 is 39
/ (slash)	Arithmetic division	39/13 is 3
^ (cover)	Power	9^2 is 81 9^(1/2) is 3
% (percent)	Percent	

An expression is an indication that an action is being performed. Expressions in MS Excel are used to write formulas and filter conditions. An expression includes at least one element, such as an operator, literal, address reference, identifier, or function. Literals, identifiers, references, and functions connected by operators called *operands*.

An operator in an expression specifies what action to perform on the operands. There are arithmetic operators, comparison operators, filtration operators, concatenation operators, identification operators and address operators. The list of operators and their purpose is given in the following table 2.1.

2.1.1 Data Types

The data type determines the format of the data stored in memory and how it is displayed in worksheet cells. The data type of the active cell or a selected block is set as follows:

- HOME/NUMBER in the window: select the type;
- Right-click on an active cell or selected range of cells and from the shortcut menu, select *Format cells* and then in the *Number tab*, select the desired type. MS Excel data types are listed in the following table 2.2.

Table 2.2

Data types

Data type	Appointment	Example
Numeric	Is used to represent numbers with a fixed decimal point	451,12 -9267,123
Currency	The numeric format used to display monetary values	451,12 R. R. 9267,12
Accounting	It is used to equalize the monetary values by the separator of the whole and fractional parts.	
Date	Used to display dates	16.4 16.4.0.42 April 02
Time	It is used to display the time	14:20 4.5.02 12:35

Data type	Appointment	Example
Percentage	The value of the cells is multiplied by 100 and is accompanied by the percent symbol	12%
Fraction	This is used to display numbers as fractions work	3/4 4/7
Scientific	Used to display floating-point numbers	1.23 E-01, here's the 1.23-digit part (mantissa) E — the base of the base (10) –01 — the order
Text	It is used to display text information. Regardless of the contents, they are treated as strings	
Special	Text, using different templates for entering specific data (phone number, postal code)	123-4567 456500
Custom	Displays text and numeric values of an arbitrary type	

2.1.2 Data entry

Any cell can be filled with data. To enter data into a cell, activate the cell (the cell is highlighted with a bold outline) and start entering. The characters you type appear immediately in the current cell and in the formula bar.

When you finish entering data into the current cell, you can do one of the following:

– <Enter> key is pressed — the data are locked in the current cell, and the selection moves one cell down;
– the tick button on the formula bar is pressed — the data will be locked in the current cell and the selection will remain in the same cell;
– any arrow key is pressed-the data are locked in the current cell and the selection moves to the cell in the direction of the arrow;

– cross button on the formula bar is pressed or the <Esc> key is pressed-data entry will be cancelled.

To enter the same data in a range of cells, you must first select the desired range of cells, then enter the necessary data and complete the input by pressing <Ctrl><Shift><Enter>.

By default, when you finish typing, the text data is aligned to the left of the cell, and the numeric data is aligned to the right. If the alignment you want to change, you need to use the command HOME/ALIGN, or right-click the mouse on the active cell or a selected range of cells and from the shortcut menu select *Format cells* and in the *Alignment tab*, select the desired type of alignment.

2.2 Entering formulas

Entering a formula must begin with the equality sign (=). A formula can include numbers, functions, references to addresses or cell names, operators, parentheses to specify the priority of operations, logical functions, and text enclosed in quotation marks. For example, =B12+A2*4 or =A1&" &B1 (the result of this formula is a Union of A1 and B1 cell values separated by a space). Let's say A1 contains the first name and B1 contains the last name. The result of the calculation of the formula will be the text containing both the first and last name. The formula, as well as the data, can be entered in several cells at once. To do this, first select the desired range of cells, then enter the desired expression and end the input by pressing <Ctrl><Shift><Enter>.

After you enter a formula, the calculated result appears in the cell, and the formula appears in the formula bar. If necessary to display a formula (not results) in the table cell, you should set the command FORMULAS / DEPENDENCIES FORMULAS / SHOW FORMULAS.

If the result of the formula calculation or format conversion is longer than the width of the column, hash symbols appear in the cell: #######. To get a numeric image, increase the width of the column.

2.3 Organization of links

Cell addresses in formulas can be placed by pointing to the appropriate cell (cell range) with the mouse. The appearance of an address reference to a cell containing the operand value depends on the selected addressing format. When you move or copy a formula, the address in the link you specify changes based on the position to which the formula is transferred. Such references are referred to as *relative references.*

Absolute references are used to enter a value from a fixed cell (whose address remains unchanged when the formula is copied or moved) into the formula. When they are denoted, a dollar sign is added to the cell address (for example, $A$20, $IA$200).

If only one address value is changed and another is committed, *mixed references* are used: the $ sign "freezes" only the column name (for example, $A9) or the row number (for example, E$6). The *F4* key is used to enter mixed and absolute references (in this case the cursor is placed either inside the reference or after it).

2.4 Autocompletion

In many tasks, you may want to fill a range of cells with an arithmetic sequence of numbers or dates. You can use one of the following methods to automatically create these sequences:

* enter data in the first two cells of the series and select them. Next, drag the fill marker (a small black square located in the lower right corner of the selected area) along the entire row. After the mouse is released, the row is filled with data;

* enter data in the first cell of the series. Pull the fill handle across the row, hold down *<Ctrl>*. The resulting sequence of numbers will always be incremented by 1;

* enter data in the first cell of the series. Select all the cells that should be filled with data. Choose *HOME/EDITING*, click the *FILL* button and choose the option *PROGRESSION.*

Next, set the type of series to be filled (usually the type is determined automatically), in the *STEP* field, specify the increment.

You can use one of *the following methods to create your own autocompletion List* for data entry:

* Set the command *FILE/OPTIONS ADVANCED* tab *GENERAL* section, then click the command button to *CHANGE* the *LISTS*. In the window *LIST ITEMS* to enter the items in this list, separating them from each other, pressing <Enter>. Click the *ADD* button.

* Enter a list in the cell range. Select the resulting range. Set the command *FILE/OPTIONS ADVANCED* tab *GENERAL* section then click the command button to *CHANGE* the *LISTS*. Make sure that the *IMPORT LIST FROM CELLS* is correct range. Click the *IMPORT* button.

To apply a List, you can use both the entire sequence of list items at once and individual items. To do this, place the cursor in the first cell of the filled range, enter the first element of the list, pull the fill marker along the entire row.

2.5 Changing the number output format

Depending on the type of table, the numeric data presented in the table may also have a different display format. To control this format, select cells with numeric information. Choose *HOME/NUMBER*, then you can:

1. Click the currency *FORMAT* is selected the number will be converted to currency format to the number of added currency symbol and two decimal places (change the monetary unit is the easiest to implement, consistently following the steps of *HOME/NUMBER*, click *CURRENCY* number *FORMAT*, enable the option *OTHER CURRENCY FORMATS* to choose another currency).

2. Click the *PERCENTAGE FORMAT* button — the selected numbers will be converted to the percentage format-the number is multiplied by 100 and the percent sign (%) is placed at the end.

3. Click the *COMMA STYLE* button — selected numbers will be formatted into a thousand every three digits of the number are separated by spaces and adds two decimal places after comma.

4. Press the button *INCREASE DECIMAL* of the formatting toolbar — the accuracy of calculations in the selected cells will increase by one digit.

5. Press the button to *DEREASE DECIMAL* of the formatting toolbar — the accuracy of calculations in the selected cells will decrease by one digit.

In general, to specify the required format you can run the command *HOME/ CELLS*, click *FORMAT*, choose *FORMAT CELLS*, then select the *NUMBER* tab and choose the desired format. To cancel the specified format and return to the usual style of the number, choose command *MAIN/STYLES*, click *CELL STYLES* and choose the *NORMAL* option.

2.6 Built-in functions of MS Excel and their applications

Built-in functions in MS Excel make it much easier to write formulas and expressions, thereby optimizing the user experience. *A function* is a subroutine that is a pre-created formula for calculating the result based on one or more arguments. In addition to the built-in functions, you can create custom functions by Visual Basic for MS Excel. Each built-in function can be identified by the name assigned to it. All built-in functions have a certain syntax by which the task values are written as function arguments (source data for calculations). The syntax of the function has a general appearance:

FUNCTION_NAME(<list of arguments>)

The arguments are entered through the character ";" (semicolon) and their syntax depends on the specific function. As arguments can be used:

* address links;
* cell or range names;
* functions;
* literals.

For example, the function allows you to sum values has the form:

1. SUM(A1:A5) — calculates the sum of all values in the range A1:A5.

2. SUM(Income) — calculates the sum of all values of the range named Income.

3. SUM(5;7;10) — calculates the sum of all values specified by the list of numbers.

The argument list is entered in parentheses after the function name without any intervals. There are functions with empty list of arguments, for example Rand(), etc.; the presence of parentheses in them is mandatory.

The following rules are used to specify a function.

First of all, you should place the cursor in the cell that should contain the result of the function. Then, you can do one of the following:

1. Press *"Shift"+"F3"*.

2. Set the command *INSERT FUNCTION* on the *FORMULAS* ribbon.

3. Click on the *fx* button in the formula bar.

Next, you should choose the category of function and the function itself in the dialog box the INSERT FUNCTION. To perform substitution step-by-step arguments using a dialog box the INSERT FUNCTION click the OK button. The following steps specify the addresses (names) of cells whose values will be used as function arguments.

> Note: if you are using another function as an argument to an input function, you must select the function name from the list of functions in the formula bar on the left and perform the required steps in the INSERT FUNCTIONS dialog box. When you are finished typing the arguments of the nested function, click the mouse pointer in the formula bar (at the end of the formula you are typing).

SUMIF(Range;Criterion;[RangeSummation]) is a mathematical function.

The function is used to sum values in cells that meet a certain criterion:

* *Range* is the range in which the criterion is defined;

* *Criterion* — specified in the form of a number, expression or text;

* *Summation range* is the range of cells to sum (optional field to fill if the *Range* and *summation range* areas are the same).

Array is a collection of ordered data of the same type. In MS Excel spreadsheet, the array is stored as a range of cells. MS Excel spreadsheet provides the ability to store a variety of information in the form of arrays, in particular vectors and matrices. This application allows you to work with arrays of different dimensions: one-dimensional, two-dimensional and three-dimensional. One-dimensional and two-dimensional arrays are placed on a single workbook sheet. Arrays of different dimensions most often contain numeric data; with which you can perform the following arithmetic operations:

– multiplying an array by a number (array elements and a single variable);

– line-by-line multiplication (elements of one-dimensional and two-dimensional arrays);

– - multiplication of arrays of any dimension.

When performing arithmetic operations of addition, multiplication, etc. over the elements of the array is necessary:

1. Select a range in accordance with the problem statement for the location of the result.

2. Use the formula bar sign " = " in accordance with the condition of the required expression.

3. Complete the expression with *Ctrl+Shift+Enter*, which is required to get the result of the task in the cell range (see the point 1).

Calculating the sum of vectors. The sequence of the following operations will calculate the sum of the vectors:

1. Enter the numeric values of vector elements in the appropriate ranges of the same dimension.

2. To calculate the result, select a range of cells of the same dimension as the original vectors.

3. Enter the following formula in the range:

= Vector_1+Vector_2.

4. To finish entering the formula with *Ctrl+Shift+Enter* (Fig.2.1).

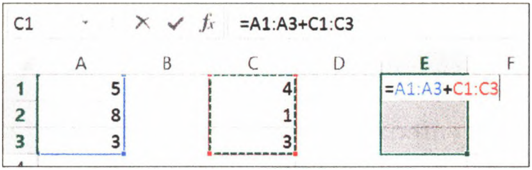


Fig. 2.1. Calculating a sum of vectors

Multiplying an array by a number. To multiply an array by a number, follow these steps:

1. Enter the numeric values of the array elements in the appropriate ranges.

2. To calculate the result, select a range of cells of the same dimension as the original array.

3. Enter the following formula in the range: = Address * 3;

4. To finish entering the formula with *Ctrl+Shift+Enter* (Fig.2.2).

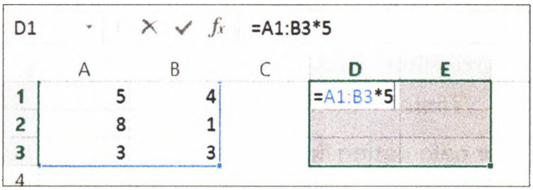


Fig. 2.2. Multiplication of an array by the number

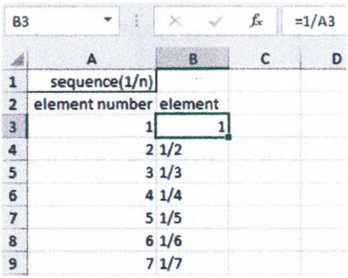
Simulation of numerical sequences and series

Numerical sequences are sets of numbers. If each number n from the natural series of numbers 1,2,3.., n ... is matched by a real number xn, then the set of numbers x1, x2, x3,..., xn ... is called a numerical sequence. The numbers x1, x2, x3, ... are called members of the sequence xn — n-s ‘ or a common element, and n is its number.

Thus, the sequence is a set of numbered elements. A sequence is specified if any of its elements is known to be retrieved. The sequence is indicated by {xn}. For example, the symbol {1/n} denotes a sequence of numbers 1, 1/2, 1/3, 1/4, ..., 1/n. In general case, to create an array of elements of the sequence, we need to perform the following steps:

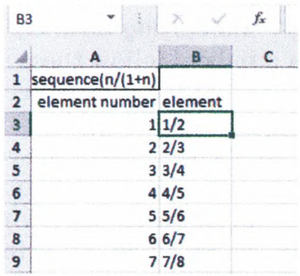
1. Create an array containing a set of positive integers;
2. To enter into a cell the sequence, making it a targeted cell references containing the numbers of elements of the sequence
3. Copy the entered formula to all other cells of the array.

An example of the creation of the sequence {1/n} and the sequence {n/(n+1)}.



B3 =1/A3

	A	B	C	D
1	sequence(1/n)			
2	element number	element		
3	1	1		
4	2	1/2		
5	3	1/3		
6	4	1/4		
7	5	1/5		
8	6	1/6		
9	7	1/7		



B3

	A	B	C
1	sequence(n/(1+n))		
2	element number	element	
3	1	1/2	
4	2	2/3	
5	3	3/4	
6	4	4/5	
7	5	5/6	
8	6	6/7	
9	7	7/8	

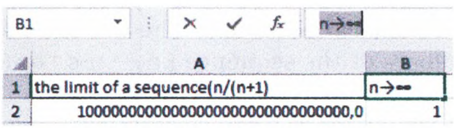
To create the most common sequences, such as arithmetic or geometric progression, the spreadsheet program has a special tool "Progression" that is:

Home → Editing → Fill → Series.

Method for calculating limits of numerical sequences

The technology of approximate calculation of the limit of the numerical sequence, consider the example. Suppose you want to find the limit of the numerical sequence

$$\lim_{n \to \infty} \frac{n}{n+1}$$



B1 n→∞

	A	B
1	the limit of a sequence(n/(n+1))	n→∞
2	10000000000000000000000000000000,0	1

Sum of a numerical series

A number series is an infinite sequence of numbers u_1, u_2, ..., u_n ... connected by a sign of summation. A series is considered to be specified if its common term $u_n = f(n)$ is known. The sum of the first n terms of the series is called the partial sum of the series. To calculate the partial sum of a series in a spreadsheet, follow these steps:

1. Calculate the first n terms of the numerical sequence
2. Calculate the sum of the terms of a numeric sequence

Numerical calculation of limits of functions

In mathematics, special techniques are used to find the limits of functions, such as the expansion of the numerator and denominator to factors, and some others. Using a spreadsheet, you can apply the following method. In a worksheet cell, enter a formula that corresponds to the functional dependency expression in which the value of the argument is specified by an address reference to the cell that contains the argument in the cell used to write the argument of the function, enter a number as close as possible to the point at which the limit of the function is calculated.

Selection of parameter

The implementation of various economic and financial projects and tasks, often requires the selection of one parameter to another parameter took the value as required. That is, if the formula calculation target is known, but the input values that allow you to get it are not known, then MS Excel uses the parameter Matching tool. This tool is a tool for solving the "What-If" data analysis problems, when the required value of the investigated function (optimality criteria) is achieved by iterating through a single value.

So, in order to determine the value that meets the required value of the optimality criterion, you need to display The tool selection parameter in the list of commands *"What-If Analysis"* in the group of tools *Data* (Fig.2.3).

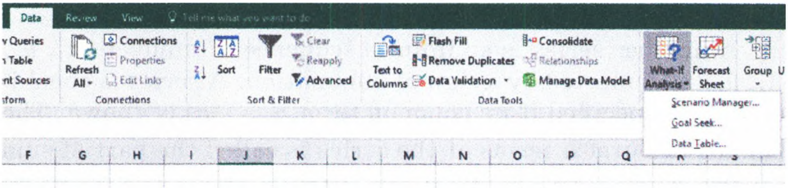


Fig. 2.3. Working with data ("What-if Analysis")

In order to apply the tool *Goal Seek* (Fig. 2.4), you need to configure the corresponding fields of the dialog box:

1. *Set cell* — link to the cell with the formula;
2. *To Value* — the target value;
3. *By changing cell* — a reference to the cell with the selected parameter;
4. Click *OK*

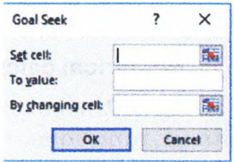


Fig. 2.4. Goal Seek dialog window

Example 2.1. It is required to determine what should be the rate of Euro, having available 3678 rubles to buy 90 euros.

Solution (the equation for this problem looks like x * 90 = 3678, where x is the desired euro rate, for which MS Excel will create a model):

1. In the cell A2, the value of the euro (previously empty cell) must be selected.
2. In cell B2, enter the formula: =A2 * 90, the preliminary result of which is 0.
3. You must specify a reference to cell B2 in the *Set cell* field, enter 3678 in the "*To value*" field, and specify a reference to cell A2 in the "*By changing cell* " value field (Fig. 2.5, answer: 40.866667)

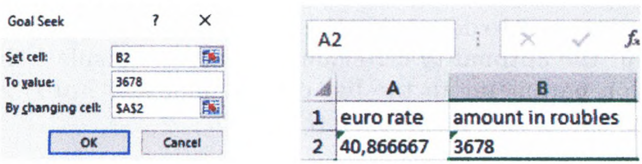


Fig. 2.5.

Example 2.2. Find the interest rate under which it is necessary to make a contribution to the bank in the amount of 500000 rubles, with a monthly compounded interest to accumulate 1000000 rubles for 4 years in the account.

Solution. The model of the problem is presented in figure 2.6. Using the built-in financial function *FV*, allowing to calculate the future value, we define its value in the conditional 10% (744 677,05 rubles).

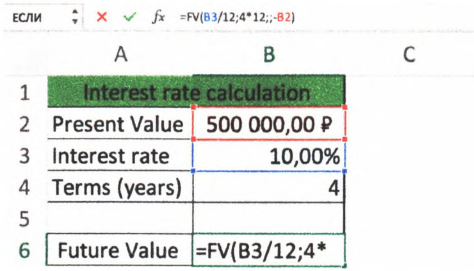


Fig. 2.6.

Steps:

1. Place the cursor in cell B6, which is the function (formula) and run the tool goal seek.

2. In the Set in cell field, there must be a reference to cell B6.

3. In the Value data box, type 1000000.

4. In the *Changing cell* value field, set a reference to cell B3 (answer: 17,45%).

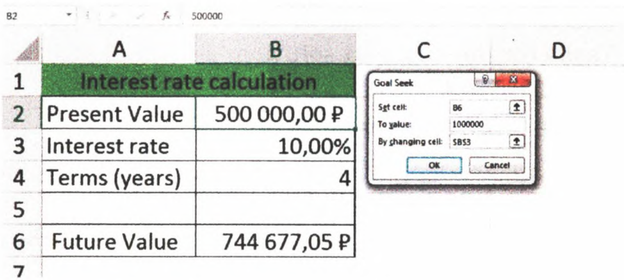


Fig. 2.7.

As a result, we get the desired value of the interest rate of 17.45% (Fig. 2.8).

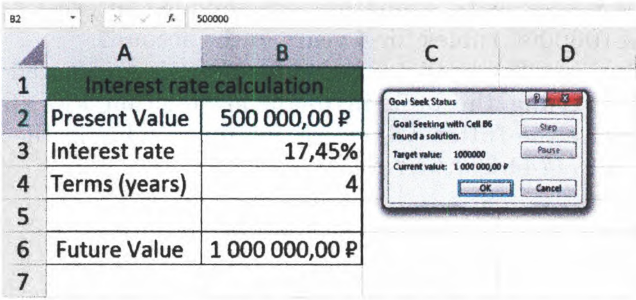


Fig. 2.8.

Thus, the parameter Selection tool selects a specific indicator, at which the desired result reaches a certain value. Now let's move to the practical section of this chapter. We are going to consider the following topics:

1. Mathematical operations in MS Excel with numbers
2. Mathematical operations in MS Excel with the values in the cells
3. Formulas in MS Excel set manually
4. Formulas in MS Excel using features
5. Calculating the values of mathematical functions in MS Excel
6. Calculation of sequence sums
7. Calculating limits of sequences
8. Selecting a parameter for the specified value in MS Excel
9. Tasks for independent solution

Instructions for students

Complete the tasks in the working file, leaving six decimal places after comma.

Write down the values in numerical and exponential (*scientific*) formats.

Problem 2.1. Perform mathematical operations using MS Excel.

1. $135 + 243$
2. 135*243
3. 135-243
4. 135/243
5. 145^3
6. $145^{\frac{2}{3}}$
7. $\sqrt[233]{145}$
8. 0,00147*(-324)
9. $\sqrt[4]{1450{,}08}$
10. -35/28,5

Put the answers in the table

Problem 2.2. To perform mathematical operations using Excel.

№	Calculate	The result, numeric format	Scientific notation
1	135+243		
2	135*243		
3	135-243		
4	135/243		
5	145^3		
6	145^(2/3)		
7	145^(1/233)		
8	0,00147*(-324)		
9	(1450,08)^(1/4)		
10	(-35)/25,5		

Problem 2.3. Perform mathematical operations using MS Excel at specified values a, b:

1. a+b
2. a * b
3. a-b
4. a / b
5. a^3
6. $a^{\frac{2}{3}}$
7. $\sqrt[233]{a}$
8. $\sqrt[233]{b}$
9. $a * (-b)$
10. $\sqrt[4]{a}$

Put answers in a table.

Problem 2.4. Perform operations using MS Excel for given values of a,b:

№	a	b	Calculate	The result, number format	The result, scientific format
1	0,332	-8,431	a+b		
2	0,332	-8,431	a*b		
3	0,332	-8,431	a-b		
4	0,332	-8,431	a/b		
5	0,332	-8,431	a^3		
6	0,332	-8,431	a^(2/3)		
7	0,332	-8,431	a^(1/233)		
8	0,332	-8,431	b^233		
9	0,332	-8,431	a*(-b)		
10	0,332	-8,431	a^(1/4)		

Problem 2.5.

1. Set the formula in MS Excel manually and find the expression value at y=34, z=10, x=243

$$\left(y + \frac{1}{z} - \frac{x}{2x + 5}\right)^{-1}$$

2. Set the formula in MS Excel manually to find the value of the expression if R=3000, n=6, i=0,12

$$R\frac{1 - e^{-n*i}}{i}$$

3. Set the formula in MS Excel manually and find the expression value at x=127, i=10, n=120, n1=40, S=100

$$x + i\frac{\frac{9n}{10} - S}{n_1}$$

4. Set the formula in MS Excel manually and find the value of the expression at n=46, k=5, ln(det R) = 34

$$-\left(n - 1 - \frac{1}{6} * (2k + 5)\right) * \ln(detR)$$

5. Set the formula in MS Excel manually and find the value of the expression at x=0,0002543

$$\frac{2x^3 - 3x + 8}{x^3 - 2x^2 + 100}$$

6. Set the formula in MS Excel manually and find the value of the expression at x=0,0002543

$$\frac{1}{\sqrt{x^2 + x} - x}$$

7. Set in MS Excel formula manually to find the value of the expression when x=0,0002543

$$2\left(\sqrt{x + \sqrt{x}} - \sqrt{x}\right)$$

8. Set in MS Excel formula manually to find the value of the expression when x=678

$$\frac{5}{25 - x}$$

9. Set in MS Excel formula manually to find the value of the expression when x=−15,25

$$\frac{x^2 - 3x + 4}{x^2 - 5x + 6}$$

10. Set in MS Excel formula manually to find the value of the expression at x=0,00025

$$\frac{(4x + 13)^3 (x + 3)}{2x + \sqrt[3]{x}}$$

Problem 2.6.

Using MS Excel functions, define formulas to calculate the following expressions and find the values

1. At x=180

$$\frac{\sin 4x}{\mathrm{tg} 2x}$$

2. At x=32

$$\left(25 \sin x + \ln(18x) - \frac{\sqrt{x}}{\mathrm{tg}(2x + 8)}\right)^{-1}$$

3. At x=0,990077

$$10x\left(\sqrt{x + \sqrt{x}} - \sqrt{x}\right)$$

4. At x=0,990077

$$2\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right)$$

5. At x=0,0002543

$$\frac{1}{\sqrt{x^2+x}-x}$$

6. At x=0

$$\frac{1}{\sqrt{x^2+x}+x}$$

7. At x=2853,006

$$\left(\frac{7-x+3x^2}{7-\operatorname{tg}5x}\right)^{\frac{2}{x}}$$

8. When x=2853,006

$$\frac{(4x+13)^3\left(x+\cos(3x-1)\right)}{2x+\sqrt[3]{x}}$$

9. When x=2853,006

$$\left(1-\frac{1}{2x}\right)^{4x-3}$$

10. When x=2853,006

$$\frac{\sin x^2}{x^2}$$

Problem 2.7.

Using MS Excel functions, define formulas to calculate the expressions and calculate them

№	formula	x	value

If, as a result, the computer generates an "error", explain its reason and specify how to change the value of the argument x.

Problem 2.8. Calculate values of functions in MS Excel.

Calculate the value of the function y(x)=k*f(x) for all values of variable x on the segment [a;b] in increments c at a given k, where f (x) is taken from the Problem 2.6.

	k	a	b	c
1	2	1	2	0,1
2	4	2	4	0,2
3	5	3	4	0,1
4	3	4	6	0,2
5	6	5	6	0,1
6	8	6	8	0,2
7	2	7	8	0,1
8	3	8	10	0,2
9	1	9	10	0,1
10	7	10	12	0,2

a	b	k
1	2	2

i	xi	f(xi) value	scientific notation

Problem 2.9.

1. Find the sum of first twenty terms of the numerical sequence

$$\sum_{n=1}^{20} \frac{5}{25 - n}$$

2. Find the sum of first 9 terms of the numerical sequence

$$\left\{n\left(n - 3\right)\right\}$$

3. Find the sum of first 30 terms of the numerical sequence

$$\left\{\sqrt[3]{n}\right\}$$

4. Find the sum of 10 to 15 terms of the numerical sequence

$$\frac{n}{\sqrt{n}}$$

5. Find the sum of first 30 terms of the numerical sequence

$$\left\{\frac{(4n+13)^3(n+\cos(3n-1))}{2n+\sqrt[3]{n}}\right\}$$

6. Find the sum of first 30 terms of the numerical sequence

$$\left\{\frac{3^{n+2}+\ln(n^7+1)+3n^6}{\sqrt[3]{4n+5}+3\lg n-3^n}\right\}$$

7. Find the sum of first 30 terms of the numerical sequence

$$\left\{\frac{2n^2+n+1}{1+2+\ldots+n}\right\}$$

8. Find the sum of first 30 terms of the numerical sequence

$$\left\{\frac{\sin n^2}{n^2}\right\}$$

9 .Find the sum of first 30 terms of the numerical sequence

$$\left\{\left(1-\frac{1}{2n}\right)^{4n-3}\right\}$$

Problem 2.10.

1. Find an approximate value of the limit

$$\lim_{n\to\infty}\frac{5}{25-n}$$

2. Find an approximate value of the limit

$$\lim_{n\to\infty} n(n-3)$$

3. Find an approximate value of the limit

$$\lim_{n\to\infty}\sqrt[3]{n}$$

4. Find an approximate value of the limit

$$\lim_{n\to\infty}\frac{n}{\sqrt{n}}$$

5. Find an approximate value of the limit

$$\lim_{n \to 2} \frac{n^2 - 3n + 4}{n^2 - 5n + 6}$$

6. Find an approximate value of the limit

$$\lim_{n \to \infty} \frac{n^3 - 3n + 4}{n^2 - 5n + 6}$$

7. Find an approximate value of the limit

$$\lim_{n \to \infty} \frac{n^2 - n + 4}{n^3 - 5n + 6}$$

8. Find an approximate value of the limit

$$\lim_{n \to \infty} \frac{2n^2 - 3n + 4}{8n^3 - 5n + 6}$$

9. Find an approximate value of the limit

$$\lim_{n \to 0} \frac{\sin n}{n}$$

10. Find an approximate value of the limit

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Problem 2.11.

1. It is known that the circumference of the first circle is 100, and the area of the second circle is 1000. Use the Goal Seek tool to determine how many times the radius of the first circle differs from the radius of the second circle.

2. The formula of a linear function is given: y=b + ax. It is known that a=10, b=20. Test the function on the range of values x from 1 to 10 with the step 1. Using Goal Seek tool by changing the value of b to determine the value of y at the point x=10, if at x=6 the value y=100.

3. The area of the first circle is 1500, the area of the second circle is 100. Use the Goal Seek tool to determine how many times the radius of the first circle differs from the radius of the second circle.

4. The formula of the linear function is given: y=2b-ax. It is known that a=25, b=10. To test the function on the interval

of values x from -3 to 5 with the increment 0,5. Using the Goal Seek tool by changing the value of b to determine the value of y at the point x=5, if at x=1 the value y= −10.

5. Find the solution of the equation $2,84x^2 - 14,7=0$.

6. Find the solution of the equation $x^2 - 11,7x + 3=0$.

7. The formula of the linear function is given: y=a−bx+3. It is known that a=10, b=20. To test the function on the interval of values x from −2 to +2 with the step 0,2. Using the Goal Seek tool by changing the value of b to determine the value of y at the point x=2, if at x=0,2 the value of y=15.

8. Find the solution of the equation $x^2 - 8,2x + 6=0$.

9. It is known that the area of the first rectangle (a1*b1) is equal to 135, and the area of the second rectangle (a2*b2) is equal to 195. Use the Goal Seek tool to determine how many times differs the a1 from a2, if b1 and b2 are equal to 3.75.

10. It is known that the area of the first right-angled triangle (a1*b1/2) is 156, and the area of the second rectangle (a2*b2/2) is 185. Using the Goal Seek tool to determine how many times a1 differs from a2, if b1 and b2 are equal to 4.15.

11. Find all roots of the equation cos(x)+sin(x)=0 in the interval [−2,5;2,5]. Draw a chart.

$$\sqrt{x^3 + 2x^2} - 5 = 0$$

12. Find all roots of the equation in the segment [−1,5;2,5]. Draw a chart.

3. Asymptotes

Approximate calculation o3f the function's behavior near points of discontinuity. Graphic construction of oblique asymptotes.

Introduction

Let $f(x)$ be defined in a neighborhood of a point a (maybe, except the point a). The existence of discontinuity at the point a means that $f(x) \neq f(a)$. Sometimes discontinuity points are also called *breakpoints*.

1. Let's suggest that $f(x)$ exists, but it is not equal to the value of $f(a)$, while the last one can exist or not, so the function cannot be defined at the point a. This situation is called *removable discontinuity*. The typical case is the uncertainty $\frac{0}{0}$, which disclosures and gets a number in the limit, for example $f(x) = \frac{\sin(\sin x)}{x}$ if x approaches zero.

2. Let's suggest that $f(x)$ does not exist, but finite single-ended limits $f(x)$ and $f(x)$, which are not equal to each other (as there is no both-sided limit). This situation is called *(unremovable) discontinuity of the 1st kind (jump)*. Typical examples of the functions with such discontinuities are the uncertainties with modulus, for instance $f(x) = \frac{|x^2 - 3x - 4|}{x + 1}$.

Sometimes the discontinuity of the 1st kind is called the discontinuity, under which finite single-ended limits exist. Then two considered types of discontinuities unite into one: the discontinuity would be removable, if single-ended limits were equal to each other, and unremovable in the other case.

3. The most difficult type of discontinuities can be considered the unremovable discontinuity of the 2nd kind (*infinity*), under which one of the single-ended limits equals infinity or does not exist. Let's note that in the case of absence of single-ended limits, like the function $f(x) = \sin\left(\sin\frac{1}{x}\right)$, additional information, which will clarify the behavior of the function in a neighborhood of the discontinuity point, is needed for the numerical investigation.

Let's take a closer look at the situation, when both of the single-ended limits equal infinity. In that case vertical line $x = a$ is called (bilateral) *vertical asymptote* of the graph of the function $f(x)$. Typical example is $f(x) = \frac{C}{(x-a)^{\alpha}}$, when $\alpha > 0$. The indicator of the power of α can be calculated through the limit $\alpha = -\frac{\ln|f(x)|}{\ln|x-a|}$. Knowing the value of α, we can find a coefficient $C = f(x)(x-a)^{\alpha}$.

Now let function $f(x)$ be defined for all sufficiently large values of $|x|$. The graph of the function is an *oblique asymptote* $y = kx + b$ on $+\infty$, if there are such k, b, so $\left(f(x) - (kx + b)\right) = 0$. An oblique asymptote on $-\infty$ is defined in the same way. In the specific case, when $k = 0$, and an oblique asymptote is a horizontal line, it is understood as *a horizontal asymptote*. The presence of asymptotes on $\pm\infty$ means that the graph of the function virtually merges with some line far away from the origin.

Coefficients of oblique asymptotes can be calculated through the formulas:

$$k = \frac{f(x)}{x},\quad b = \left(f(x) - kx\right).$$

Practice:

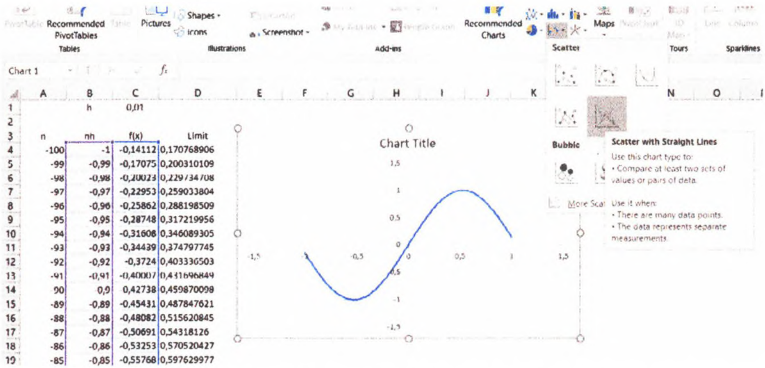
1. Let's calculate the table of values for the function $f(x) = \frac{\sin\sin 3x}{x^2 + 2x}$ if $x = nh$, where $h = 0.01$,

$n = -100, -99, \ldots, -1, 1, 2, \ldots, 100$. Let's draw two lines according to the found points: separately for the negative and positive n. Let's find the numerical value of the limit $f(x)$.

1.1. Enter into A4:A203 in the Excel sheet numbers $-100, -99, \ldots, -1, 1, 2, \ldots, 100$. Enter 0,01 into the cell C1.

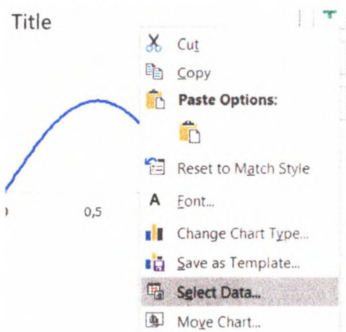
1.2. Enter formula =A4*$C$1 into the cell B4. Drag and drop the formula till the cell B203.

1.3. Highlight B4:C103. Set the command *"Insert/Charts"* and choose the type *"Scatter with Straight Lines"* as represented in the figure:



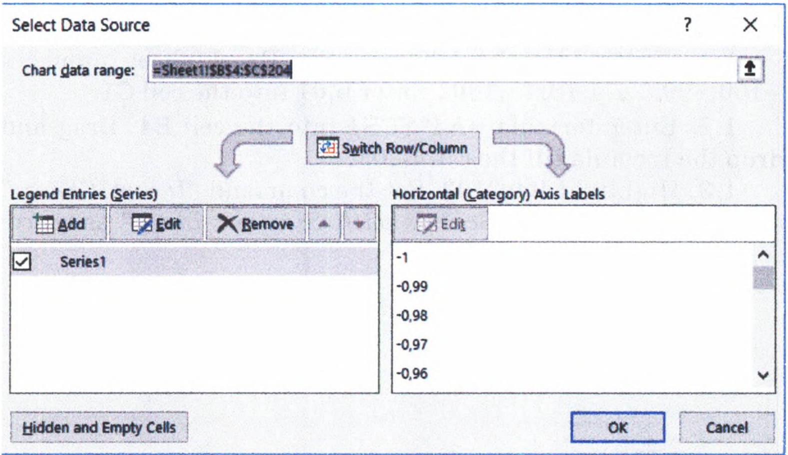
1.4. Draw up the table as it is shown in the figure above.

1.5. Highlight the chart.

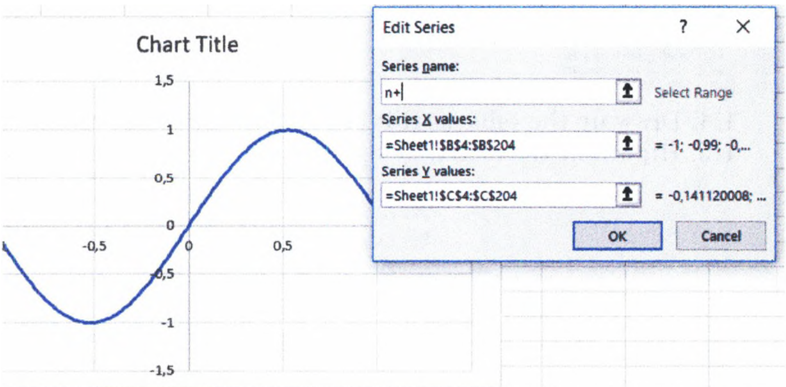


1.6. Select the command *"Select Data"* in the contextual menu.

1.7. Set the command *"Add"* in the window *"Select Data Source"*.



1.8. Fill the information in the window *"Edit"* as it is shown in the figure:



1.9. In the end, we will get the polyline like in the picture.

1.10. Enter the formula to calculate the limit $f(x)$ into the cell D4:

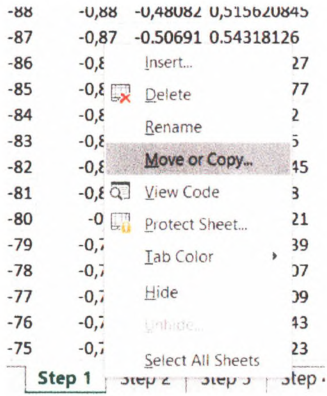
=SIN(3*(B4+$C$1))/((B4+$C$1)^2+2*(B4+$C$1))

1.11. Drag and drop this formula till the cell D203.

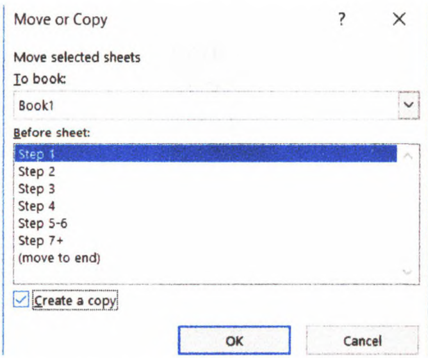
1.12. Name the sheet *Step 1*.

2. Repeat the same actions for $h = 0.0001$. Clarify the value of the limit.

2.1. Create a copy for the sheet *Step 1*. For doing this we need to choose the command "Move or Copy..." in the contextual menu of the sheet *Step 1* (see the figure).



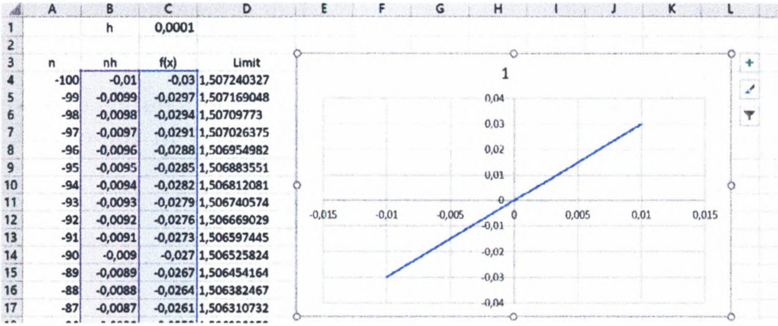
2.2. Draw up the window "Move or Copy..." as it is shown in the figure:



2.3. Name the new sheet as *Step 2*.

2.4. In the sheet *Step 2* change the value of the cell C1 to 0,0001.

2.5. As a result, the data should be presented in the sheet in the same way as in the figure.



	A	B	C	D
1		h	0,0001	
2				
3	n	nh	f(x)	Limit
4	-100	-0,01	-0,03	1,507240327
5	-99	-0,0099	-0,0297	1,507169048
6	-98	-0,0098	-0,0294	1,50709773
7	-97	-0,0097	-0,0291	1,507026375
8	-96	-0,0096	-0,0288	1,506954982
9	-95	-0,0095	-0,0285	1,506883551
10	-94	-0,0094	-0,0282	1,506812081
11	-93	-0,0093	-0,0279	1,506740574
12	-92	-0,0092	-0,0276	1,506669029
13	-91	-0,0091	-0,0273	1,506597445
14	-90	-0,009	-0,027	1,506525824
15	-89	-0,0089	-0,0267	1,506454164
16	-88	-0,0088	-0,0264	1,506382467
17	-87	-0,0087	-0,0261	1,506310732

3. Compose a table of values for the function $f(x) = \dfrac{1}{1 + 2^{\frac{1}{x}}}$ if $x = nh$, where $h = 0.01$, $n = -100, -99, \ldots, -1, 1, 2, \ldots, 100$. Draw two lines according to the found points: separately for negative and positive n. Find the numerical value for the single-ended limits $f(x), f(x)$.

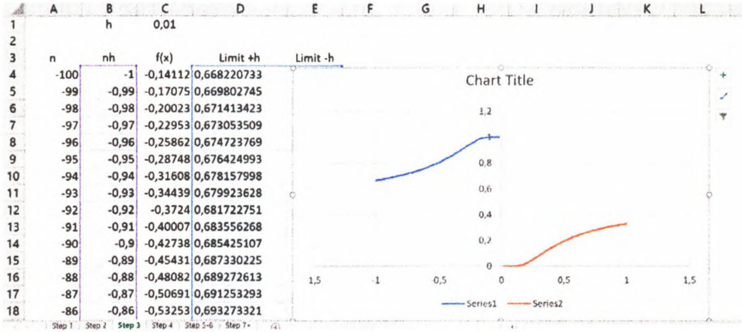
3.1. Create a new sheet and name it *Step 3*.

3.2. Enter into A4:A203 of the Excel sheet numbers $-100, -99, \ldots, -1, 1, 2, \ldots, 100$. Enter 0,01 into the cell C1.

3.3. Enter the formula =A4*$C$1 into the cell B4. Drag and drop the formula till the cell B203.

3.4. Draw two lines according to the found points: separately for negative and positive n, as it is instructed in paragraphs above.

3.5. In the end, there should be two lines as it is shown in the figure:



	A	B	C	D	E
1		h	0,01		
2					
3	n	nh	f(x)	Limit +h	Limit -h
4	-100	-1	-0,14112	0,668220733	
5	-99	-0,99	-0,17075	0,669802745	
6	-98	-0,98	-0,20023	0,671413423	
7	-97	-0,97	-0,22953	0,673053509	
8	-96	-0,96	-0,25862	0,674723769	
9	-95	-0,95	-0,28748	0,676424993	
10	-94	-0,94	-0,31608	0,678157998	
11	-93	-0,93	-0,34439	0,679923628	
12	-92	-0,92	-0,3724	0,681722751	
13	-91	-0,91	-0,40007	0,683556268	
14	-90	-0,9	-0,42738	0,685425107	
15	-89	-0,89	-0,45431	0,687330225	
16	-88	-0,88	-0,48082	0,689272613	
17	-87	-0,87	-0,50691	0,691253293	
18	-86	-0,86	-0,53253	0,693273321	

3.6. Draw up the table as it is shown in the picture.

3.7. Enter the formula for calculating the numerical value of the limit $f(x)$ into the cell D4:

=1/(1+2^(1/(B4+$C$1)))

3.8. Enter the formula for calculating the numerical value of the limit $f(x)$ into the cell D4:

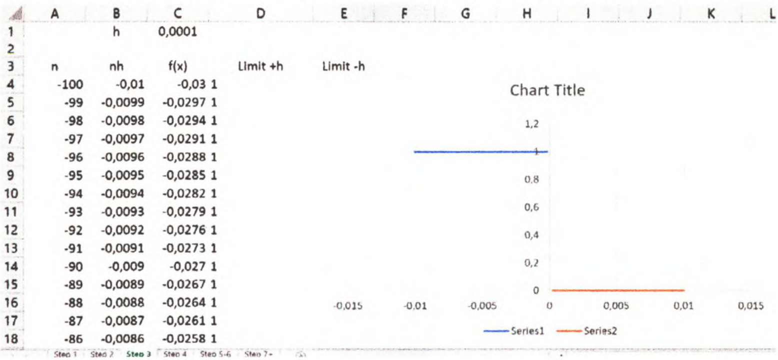
=1/(1+2^(1/(B4-$C$1))).

4. Repeat these actions for $h = 0.0001$. Clarify the values of limits.

4.1 Create the copy of the sheet *Step 3* as it is instructed in paragraphs 2.1.-2.4. Name the new sheet *Step 4*.

4.2. Change the value of the cell C1 to 0,0001 in the sheet *Step 4*.

4.3. Finally, there should be data as in the figure.



	A	B	C	D	E
1		h	0,0001		
2					
3	n	nh	f(x)	Limit +h	Limit -h
4	-100	-0,01	-0,03	1	
5	-99	-0,0099	-0,0297	1	
6	-98	-0,0098	-0,0294	1	
7	-97	-0,0097	-0,0291	1	
8	-96	-0,0096	-0,0288	1	
9	-95	-0,0095	-0,0285	1	
10	-94	-0,0094	-0,0282	1	
11	-93	-0,0093	-0,0279	1	
12	-92	-0,0092	-0,0276	1	
13	-91	-0,0091	-0,0273	1	
14	-90	-0,009	-0,027	1	
15	-89	-0,0089	-0,0267	1	
16	-88	-0,0088	-0,0264	1	
17	-87	-0,0087	-0,0261	1	
18	-86	-0,0086	-0,0258	1	

5. Calculate the table of values for the function $f(x) = \dfrac{\sin\sin 2x}{\sqrt[3]{x^5 + 3x^4}}$ if $x = nh$, where $h = 0.01$, $k = -100, -99, \ldots, -1, 1, 2, \ldots, 100$. Draw two lines according to the found points: separately for negative and positive n.

5.1. Create a new sheet and name it *Step 5-6*.

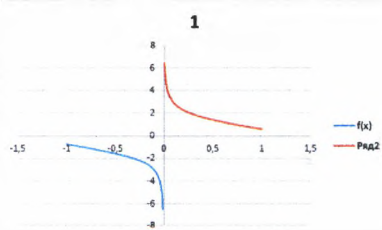
5.2. Enter into A4:A203 of the new Excel sheet numbers $-100, -99, \ldots, -1, 1, 2, \ldots, 100$. Enter 0,01 into the cell C1.

5.3. Enter the formula =A4*$C$1 into the cell B4. Drag and drop that formula till the cell B203.

5.4. Draw two lines according to the found points: separately for negative and positive n, as it is instructed in paragraphs 1.3–1.8.

5.5. In the result, there should be two polylines as it is shown in the figure:

	A	B	C	D	E	F
1		h	0,01			
2						
3	n	nh	f(x)	ln|f(x)|/ln|x|	f(x)*x^α	C/(x^α)
4	-100	-1	-0,72171			
5	-99	-0,99	-0,73677	-30,39514855	-0,733779	1,001841
6	-98	-0,98	-0,75189	-14,11502855	-0,7457723	1,005959
7	-97	-0,97	-0,76708	-8,705494066	-0,757687	1,010137
8	-96	-0,96	-0,78233	-6,013259355	-0,7695203	1,014375
9	-95	-0,95	-0,79765	-4,407679744	-0,7812693	1,018676
10	-94	-0,94	-0,81303	-3,345216754	-0,7929313	1,023041
11	-93	-0,93	-0,82847	-2,59293316	-0,8045036	1,027472
12	-92	-0,92	-0,84398	-2,034366789	-0,8159834	1,03197
13	-91	-0,91	-0,85955	-1,604821817	-0,827368	1,036538
14	-90	-0,9	-0,87517	-1,265485309	-0,8386546	1,041176
15	-89	-0,89	-0,89087	-0,991662028	-0,8498405	1,045888
16	-88	-0,88	-0,90662	-0,766891855	-0,8609231	1,050674
17	-87	-0,87	-0,92243	-0,579782265	-0,8718997	1,055537
18	-86	-0,86	-0,93831	-0,422198318	-0,8827675	1,06048



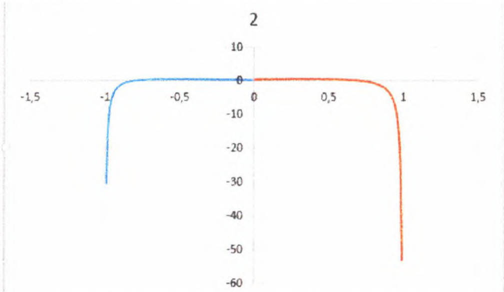
6. Calculate the values of $\dfrac{\ln|f(x)|}{\ln|x|}$ in the same points of x for the function which is investigated and, after drawing two appropriate lines, estimate the value of α.

6.1. In the sheet *Step 5-6* enter the formula into the cell D5 for calculating values of $\dfrac{\ln|f(x)|}{\ln|x|}$:

=-LN(ABS(C5))/LN(ABS(B5))

6.2. Drag and drop that formula till the cell D202.

6.3. After highlighting the appropriate ranges of cells (for negative n B4:B103, D4:D103; for positive n B104:B203, D104:D202) draw two lines according to the found points: separately for negative and positive n, as it is instructed in paragraphs 1.3–1.8. In the end, there should be two lines as it is shown in the figure:



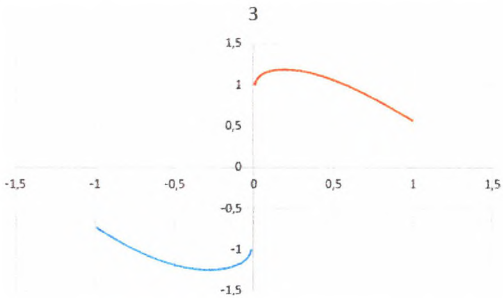
6.4. Take the value of α from the cell D103 which equals -0,404555323102164.

7. Calculate values of $f(x)x^{\alpha}$ in the same points x for the found α and the function, which is investigated, and estimate the value of C after drawing two appropriate lines.

7.1. Enter the formula into the cell E5 in the sheet Step 5-6 for calculating $f(x)x^{\alpha}$:

=C5*ABS(B5)^$D$103

7.2. After highlighting the appropriate ranges of cells (for negative n B4:B103, E4:E103; for positive n B104:B203, E104:E202) draw two lines according to the found points: separately for negative and positive n, as it is instructed in previous steps. Finally, there should be two lines as it is shown in the figure:



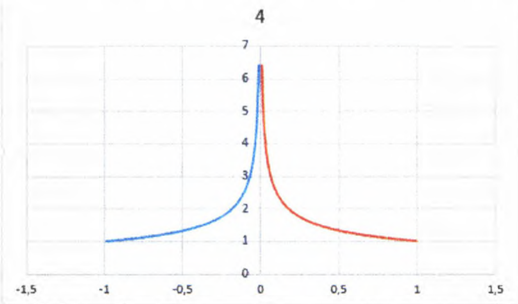
7.3. Take the value of C from the cell E102 which equals 1,02492234237999.

8. Draw two branches of the graph $y = \dfrac{C}{x^{\alpha}}$ according to the same array of points x and compare it with the graph of the original function.

8.1. Enter the formula into the cell F5 in the sheet *Step 5-6* for calculating the value of $y = \dfrac{C}{x^{\alpha}}$:

=$E$104/(ABS(B5)^$D$104)

8.2. After highlighting the appropriate ranges of cells (for negative n B4:B103, F4:F103; for positive n B104:B203, F104:F202) draw two lines according the found points: separately for negative and positive n, as it is instructed in previous points. Finally, there should be two curves as it is shown in the figure:



9. Repeat steps 5-8 for $h = 0.0001$, while clarifying the values of all limits.

9.1. Create a copy of the *Step 5-6* as it is instructed in previous steps. Name the new sheet *Step 7*.

9.2. Change the value of the cell C1 to 0,0001 in the sheet *Step 7*.

10. Calculate the table of values for the function $f(x) = (x+2)\text{arctg}3x$ if $x = nh$, where $h = 0.01$, $n = -100,-99,\ldots,-1,1,2,\ldots,100$. Draw two lines according to the found points: separately for negative and positive n.

10.1. Create a new sheet and name it *Step 8.*

10.2. Enter into the range of cells A4:A203 in the new Excel sheet numbers $-100, -99, \ldots, -1, 1, 2, \ldots, 100$. Enter number 1 into the cell C1.

10.3. Enter the formula =A4*$C$1 into the Cell B4. Drag and drop this formula till the cell B203..

10.4. Enter the formula into the cell C5 for calculating the values of $f(x) = (x+2)\text{arctg}3x$:

=(B4+2)*ATAN(3*B4)

10.5. Draw two lines according to the found points: separately for negative and positive *n*.

11. Calculate the values of $\frac{f(x)}{x}$ in the same points x for the function which is investigated and, after drawing two appropriate lines, estimate the value of k.

11.1. Enter the formula for calculating the value of $\frac{f(x)}{x}$ into the cell D4 in the sheet *Step 8*:

=C4/B4

11.2. Draw two lines according to the found points: separately for negative and positive *n*.

11.3. For the negative infinity take the value of k which equals 1,24904577239825 (from the cell D4) and for the positive infinity 3,74713731719476 (from the cell D203).

12. Calculate values of $f(x) - kx$ in the same points x for the found k and the function which is investigated, and after drawing two appropriate lines, estimate the values of b.

12.1. Enter the formula for calculating values of $f(x) - kx$ for the negative infinity in the cell E4 in the sheet *Step 8*:

=C4-$D$4*B4.

Copy the formula till the cell E103.

12.2. Enter the formula for calculating values of $f(x) - kx$ for the positive infinity in the cell E104 in the sheet *Step 8*:

=C4-$D$4*B4.

Copy the formula till the cell E203.

12.3. Draw two lines according to the found points: separately for negative and positive *n*.

12.4. For the negative infinity take the value of b which equals 1,27119062808401 (from the cell D5) and for the positive infinity 3,76322770096158 (from the cell D202).

13. Draw two oblique asymptotes of the graph $y = kx + b$ according to the same array of points x and compare it with the graph of the original function.

13.1. Enter the formula for calculating values of $y = kx + b$ for the negative infinity in the cell F4 in the sheet *Step 8*:

=$D$4*B4+$D$5.

Copy the formula till the cell F103.

13.2. Enter the formula for calculating values of $y = kx + b$ for the positive infinity in the cell F104 in the sheet *Step 8*:

=$D$4*B4+$D$202.

Copy the formula till the cell F203.

13.3. Draw two lines according to the found points: separately for negative and positive *n*.

14. Repeat the instructions of paragraphs 10-13 for *h*=10, clarifying values of limits.

14.1. Create a copy of the sheet *Step 8* as it is instructed in paragraphs 2.1-2.4. Name it *Step 9*.

14.2. Change the value in the cell C1 to 10 in the sheet *Step 9*.

Exercises for independent work

1. Study the points of discontinuity of the functions by MS Excel:

a) $f(x) = \dfrac{\operatorname{arctg}(x^2 + x)}{x^2 + 4x + 3}$, c) $f(x) = \dfrac{2^{x+2} - 1}{\sqrt{x^2 + 4x + 4}}$.

b) $f(x) = \dfrac{\ln(x^2 + 3x + 1)}{\sqrt[5]{(x^2 + 2x - 3)^6}}$,

2. Find oblique asymptotes for the functions by MS Excel:

a) $f(x) = \sqrt[4]{x^4 + 5x^3 - 1}$, b) $f(x) = \dfrac{x^2 \operatorname{arctg}(2x - 1)}{x - 2}$.

4. Derivatives

Approximate calculation of a derivative at the given point

Introduction

Let the function $f(x)$ be defined in a neighborhood of some point a. Then the *derivative* of the function $f(x)$ in the point a is $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

The simplest formulas for an approximate numerical calculation of the derivative are so called *two-point* formulas, which can be extracted straightforwardly from the definition of the derivative: $f'(a) \approx \dfrac{f(a+h) - f(a)}{h}$, $f'(a) \approx \dfrac{f(a) - f(a-h)}{h}$. This name is attributed to the fact that these formulas allow to determine the value of a derivative in a point by values of a function in two other points.

If we trace the function and point a and explore the dependence of measure of inaccuracy of two-point derivative evaluation from low increment h, it is possible to prove that the measure of inaccuracy will be proportionate to the magnitude of h.

It is possible to obtain more accurate estimation if we use values of the function of three points:

$$f'(a) \approx \frac{-f(a+2h) + 4f(a+h) - 3f(a)}{2h},$$

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h},$$

$$f'(a) \approx \frac{3f(a) - 4f(a-h) + f(a-2h)}{2h}.$$

The first formula is used for calculation of a derivative on the left boundary of an interval, the second formula is used for evaluation of a derivative in all inner points of the interval, and the third formula is used to calculate a derivative on the right boundary of an interval. Measures of inaccuracy of three-point formulas are proportionate to h^2.

Formulas that are even more accurate involve derivative's evaluation basing on bigger amount of points (4 and more). However, it is important to take into account that the increase in number of points that take part in calculation makes it more complicated and enhance the rounding error, which occurs during these calculations. That is why it is never done in everyday practice, although theoretically it is possible to use the whole array of known function values.

Another way of making the value of a derivative more accurate is to diminish the pitch h. This approach is more rational for functions defined by analytical expression, however, it cannot be used for those functions, which are defined by a table of values.

Example 4.1. Calculate a derivative of function at a given point using the following algorithm in MS Excel.

Algorithm for finding a derivative.

1. Compile the table of function values $f(x) = \dfrac{x^2 + 3x + 1}{x^2 + 2x + 2}$ with $x = nh$, where $h = 1$, $n = -10,-9,\dots,10$.

1.1. Enter numbers from -10 till 10 with pitch 1 into cell range A3:A23 of Excel worksheet, as shown in the picture. Enter into cell C1 number 1. Enter into cell B3 the formula for finding function's values $f(x) = \dfrac{x^2 + 3x + 1}{x^2 + 2x + 2}$ with $x = nh$, where $h = 1$, $n = -10,-9,\dots,10$:

=(A3^2+3*A3+1)/(A3^2+2*A3+2)

1.2. Copy the formula into the cell B23.

2. In every point $x = nh$ with $n = -10,-9,\dots,9$ estimate the value of the derivative by two-point formula

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

Draw the polyline.

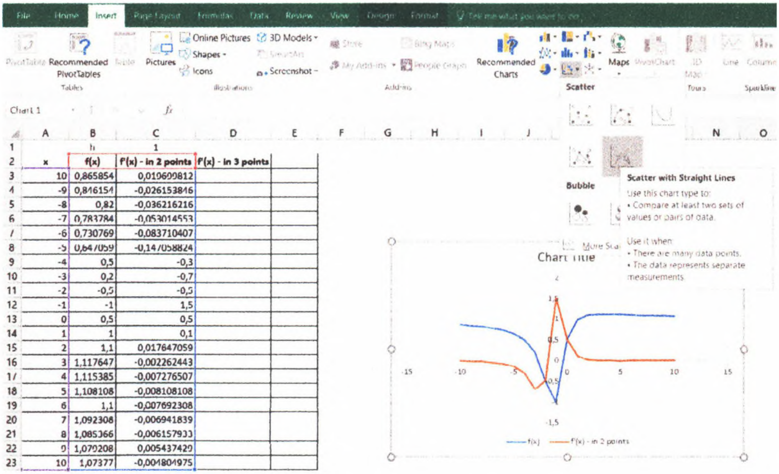
2.1. Enter into cell C3 following formula:

=(((A3+$C$1)^2+3*(A3+$C$1)+1)/
((A3+$C$1)^2+2*(A3+$C$1)+2)-((A3)^2+3*(A3)+1)/
((A3)^2+2*(A3)+2))/$C$1

2.2. Copy the formula to the cell C23.

2.3. To draw the polygonal line, enter into cells A2, B2, C2 respectively x, f(x), f′(x) — in 2 points, as shown in the picture.

2.4. Select cells A2:C23. Make a command INSERT/ CHARTS and choose the diagram type *Scatter with Straight Lines*, as shown in the figure:



2.5. Press OK.

2.6. As result, we will obtain a polyline as shown in the picture.

3. In every point $x = nh$ where $n = -10, -9, \ldots, 10$ estimate the value of the derivative of the function using three-point formula. Draw a polynomial line.

3.1 Into cell D3 enter the formula:

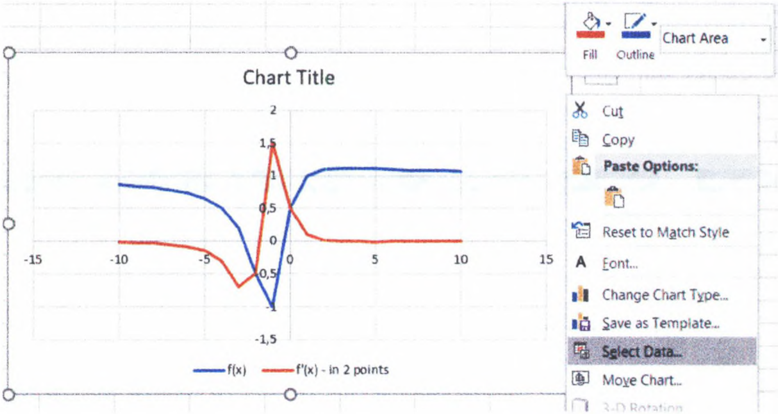
=(((A3+$C$1)^2+3*(A3+$C$1)+1)/
((A3+$C$1)^2+2*(A3+$C$1)+2)-
((A3-$C$1)^2+3*(A3-$C$1)+1)/
((A3-$C$1)^2+2*(A3-$C$1)+2))/(2*$C$1)

3.2. Copy the formula till cell D23.

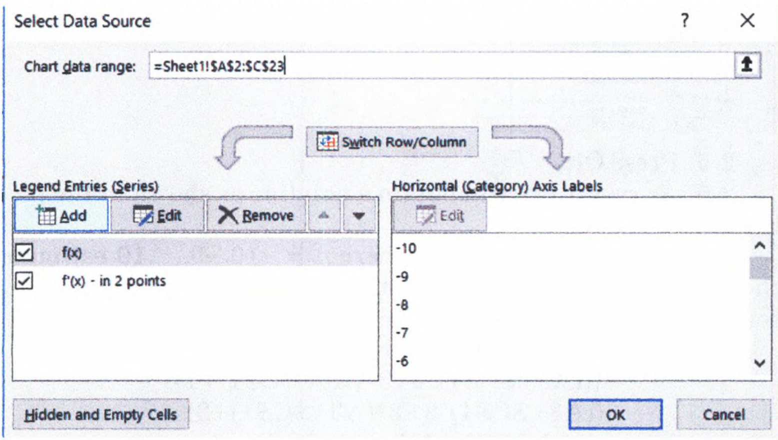
3.3. To build a polynomial line into cell D2 enter f′(x) — in 3 points, as shown in the figure.

3.4. Select the diagram.

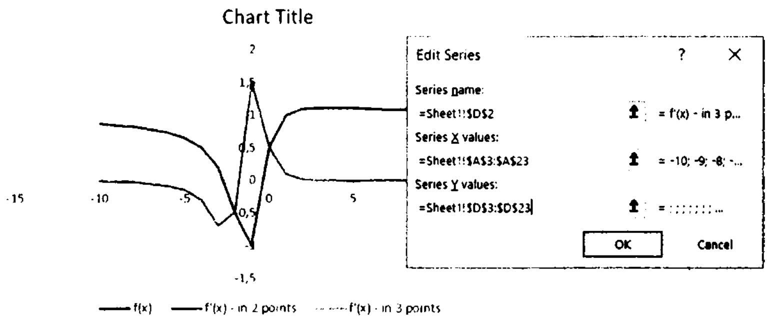
3.5. Choose the command “Select Data...” from context menu.



3.6. Choose command *Add* in window *Select Data Source.*



3.7. Fill in the window *Edit Series* as shown in the figure:



3.8. As a result, we obtain the polyline as shown in the figure.

4. Compile the table of accurate values of a derivative $f'(x) = \dfrac{-x^2 + 2x + 4}{\left(x^2 + 2x + 2\right)^2}$ with the same values of x.

4.1. Enter a formula into cell E3:

=(-(A3^2)+2*A3+4)/((A3^2+2*A3+2)^2)

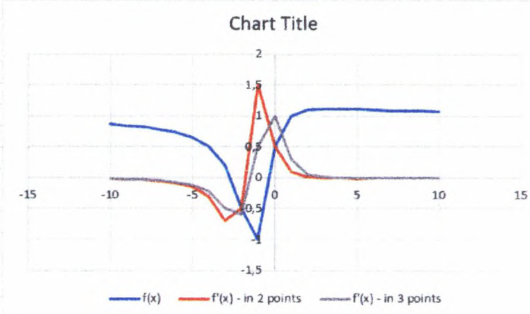
4.2. Copy till cell E23.

4.3.To build a polynomial line into cell E2 enter f′(x) point as shown in the figure.

4.4. Add new data from the range of cells E3:E23 as shown in point 3 above.

4.5. As a result, we will obtain a polynomial line as shown in the figure (on page 62).

1		h	1		
2	x	f(x)	f'(x) - in 2 points	f'(x) - in 3 points	f'(x) - point
3	-10	0,865854	-0,019699812	-0,017517136	-0,0172516
4	-9	0,846154	-0,026153846	-0,022926829	-0,0224852
5	-8	0,82	-0,036216216	-0,031185031	-0,0304
6	-7	0,783784	-0,053014553	-0,044615385	-0,0430972
7	-6	0,730769	-0,083710407	-0,06836248	-0,0650888
8	-5	0,647059	-0,147058824	-0,115384615	-0,1072664
9	-4	0,5	-0,3	-0,223529412	-0,2
10	-3	0,2	-0,7	-0,5	-0,44
11	-2	-0,5	-0,5	-0,6	-1
12	-1	-1	1,5	0,5	1
13	0	0,5	0,5	1	1
14	1	1	0,1	0,3	0,2
15	2	1,1	0,017647059	0,058823529	0,04
16	3	1,117647	-0,002262443	0,007692308	0,00346021
17	4	1,115385	-0,007276507	-0,004769475	-0,0059172
18	5	1,108108	-0,008108108	-0,007692308	-0,0080351
19	6	1,1	-0,007692308	-0,007900208	-0,008
20	7	1,092308	-0,006941839	-0,007317073	-0,0073373
21	8	1,085366	-0,006157933	-0,006549886	-0,0065437
22	9	1,079208	-0,005437429	-0,005797681	-0,0057837
23	10	1,07377	-0,004804975	-0,005121202	-0,0051062

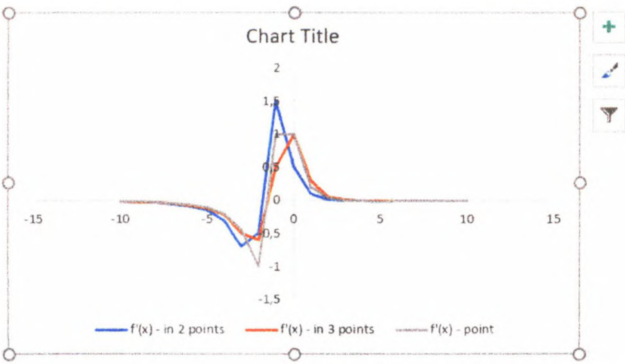


5. Compare the accurate values of the derivatives with approximate values building three polylines in one picture.

5.1. Select cell range A2:A23, C2:E23 as shown in the picture:

A	B	C	D	E
	h	1		
x	f(x)	f'(x) - in 2 points	f'(x) - in 3 points	f'(x) - point
-10	0,865854	-0,019699812	-0,017517136	-0,0172516
-9	0,846154	-0,026153846	-0,022926829	-0,0224852
-8	0,82	-0,036216216	-0,031185031	-0,0304
-7	0,783784	-0,053014553	-0,044615385	-0,0430972
-6	0,730769	-0,083710407	-0,06836248	-0,0650888
-5	0,647059	-0,147058824	-0,115384615	-0,1072664
-4	0,5	-0,3	-0,223529412	-0,2
-3	0,2	-0,7	-0,5	-0,44
-2	-0,5	-0,5	-0,6	-1
-1	-1	1,5	0,5	1
0	0,5	0,5	1	1
1	1	0,1	0,3	0,2
2	1,1	0,017647059	0,058823529	0,04
3	1,117647	-0,002262443	0,007692308	0,00346021
4	1,115385	-0,007276507	-0,004769475	-0,0059172
5	1,108108	-0,008108108	-0,007692308	-0,0080351
6	1,1	-0,007692308	-0,007900208	-0,008
7	1,092308	-0,006941839	-0,007317073	-0,0073373
8	1,085366	-0,006157933	-0,006549886	-0,0065437
9	1,079208	-0,005437429	-0,005797681	-0,0057837
10	1,07377	-0,004804975	-0,005121202	-0,0051062

5.2. Command Insert/ Charts and choose the type of a diagram *Scatter with straight lines*. As a result, we should obtain diagram as in the figure:



6. Repeat calculations of steps 1-5 using the increment $h = 0.1$.

6.1. For this purpose, compile a table on the new worksheet as shown in the figure:

	A	B	C	D	E	F
1			h	0,1		
2	x	nh	f'(x)	f'(x) - в 2-х т.	f'(x) - в 3-х т.	f'(x) точ
3	-10	-1	-0,02704	1,188118812	0,99009901	1
4	-9	-0,9	-0,02785	1,50418888	1,346153846	1,362612
5	-8	-0,8	-0,02805	1,711362032	1,607775456	1,627219
6	-7	-0,7	-0,0269	1,803226827	1,75729443	1,775945
7	-6	-0,6	-0,02262	1,793103448	1,798165138	1,813317
8	-5	-0,5	-0,01065	1,705882353	1,749492901	1,76
9	-4	-0,4	0,022624	1,569285432	1,637583893	1,643599
10	-3	-0,3	0,126697	1,407759044	1,488522238	1,490924
11	-2	-0,2	0,520362	1,239725104	1,323742074	1,323617
12	-1	-0,1	1,447964	1,077348066	1,158536585	1,156863
13	0	0	0,927602	0,92760181	1,002474938	1
14	1	0,1	0,452489	0,793709665	0,860655738	0,857886
15	2	0,2	0,266968	0,676458041	0,735083853	0,73233
16	3	0,3	0,180995	0,575203456	0,625830749	0,623264
17	4	0,4	0,134006	0,488565489	0,531884472	0,529584
18	5	0,5	0,105173	0,414866033	0,451715761	0,449704
19	6	0,6	0,085973	0,352387279	0,383626656	0,381896
20	7	0,7	0,072398	0,299510113	0,325948696	0,324476
21	8	0,8	0,062355	0,254778374	0,277144243	0,275899
22	9	0,9	0,054657	0,21691974	0,235849057	0,2348
23	10	1	0,048587	0,184842884	0,200881312	0,2

6.2. The following formula is entered in the cell B3:

=A3*$D$1

6.3. The following formula is entered in the cell C3:

=(((A3)^2+3*A3+1)/(A3^2+2*A3+2)-($D$1^2+3*$D$1+1)/($D$1^2+2*$D$1+2))/(A3-$D$1)

6.4. The following formula is entered in cell D3:

=(((B3+$D$1)^2+3*(B3+$D$1)+1)/((B3+$D$1)^2+2*(B3+$D$1)+2)-((B3)^2+3*(B3)+1)/((B3)^2+2*(B3)+2))/$D$1

6.5. The following formula is entered in the cell E3:

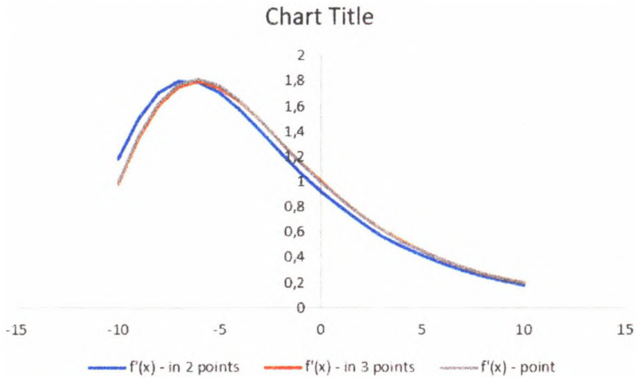
=(((B3+$D$1)^2+3*(B3+$D$1)+1)/((B3+$D$1)^2+2*(B3+$D$1)+2)-((B3-$D$1)^2+3*(B3-$D$1)+1)/((B3-$D$1)^2+2*(B3-$D$1)+2))/(2*$D$1)

6.6. The following formula is entered in the cell F3:

=(-(B3^2)+2*B3+4)/((B3^2+2*B3+2)^2)

6.7. Select cell range A2:A23, D2:F23.

6.8. Command INSER/CHARTS and select type of a diagram *Scatter with straight lines*. As a result, we should obtain diagram as shown in the picture:



7. Compile the table inaccuracy measures for the derivative calculated by two-point method through finding difference $f'(0) - \dfrac{f(h) - f(0)}{h}$ with $h = n\delta$, where $\delta = 0.01$, $n = -50, -49, \ldots, -1, 1, 2, \ldots, 50$.

7.1. Using a new worksheet, enter into cell range A4:A103 numbers -50, -49,...,-1,1,2,...,50 with step 1 as shown in the figure (beginning of the table):

	A	B	C	D	E
1			0		0,01
2					
3	n	h	2 points	3 points	
4	-50	-0,5	-0,4	-0,046153846	
5	-49	-0,49	-0,39275	-0,044964962	
6	-48	-0,48	-0,38539	-0,043748374	
7	-47	-0,47	-0,37794	-0,042507243	
8	-46	-0,46	-0,37039	-0,041244682	
9	-45	-0,45	-0,36276	-0,039963747	

	A	B	C	D	E
3	n	h	2 points	3 points	
10	-44	-0,44	-0,35505	-0,038667437	
11	-43	-0,43	-0,34727	-0,037358693	
12	-42	-0,42	-0,33942	-0,036040393	
13	-41	-0,41	-0,3315	-0,034715354	
14	-40	-0,4	-0,32353	-0,033386328	
15	-39	-0,39	-0,3155	-0,032055998	
16	-38	-0,38	-0,30743	-0,030726985	
17	-37	-0,37	-0,29931	-0,029401838	
18	-36	-0,36	-0,29115	-0,028083038	
19	-35	-0,35	-0,28295	-0,026772997	
20	-34	-0,34	-0,27473	-0,025474055	
21	-33	-0,33	-0,26648	-0,024188483	
22	-32	-0,32	-0,25821	-0,022918481	
23	-31	-0,31	-0,24992	-0,021666175	
24	-30	-0,3	-0,24161	-0,020433622	
25	-29	-0,29	-0,2333	-0,019222808	
26	-28	-0,28	-0,22497	-0,018035646	
27	-27	-0,27	-0,21665	-0,016873979	
28	-26	-0,26	-0,20832	-0,015739578	
29	-25	-0,25	-0,2	-0,014634146	

7.2 Enter number 0 into cell C1. Enter number 0,01 into the cell E1.

7.3. Enter the following formula into the cell B4:

=A4*$E$1

7.4. Enter the following formula into the cell C4:

=(-$C$1^2+2*$C$1+4)/(($C$1^2+2*$C$1+2)^2)-((B4^2+3*B4+1)/(B4^2+2*B4+2)-($C$1^2+3*$C$1+1)/($C$1^2+2*$C$1+2))/B4

8. Explore the dependence of measure of inaccuracy from *h*, building a plot using data from the previous table.

8.1. Select cell range A3:A103, C3:C103.

8.2. Command INSERT/CHARTS and select the type of a diagram *Scatter with straight lines.* As a result, we will obtain the diagram as shown in the picture (see point 9).

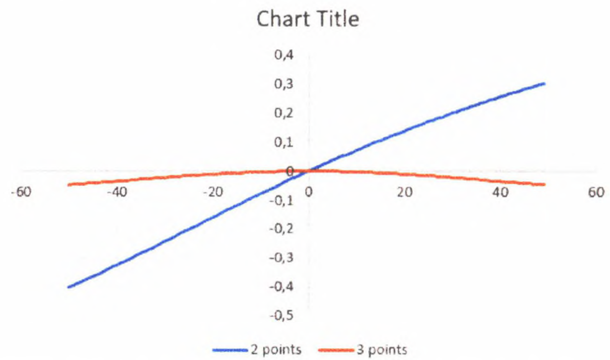
9. Repeat steps 7-8 for calculation the measure of inaccuracy for three-point formula $f'(0) = \dfrac{f(h) - f(-h)}{2h}$.

9.1 Enter the following formula into the cell D4:

=(-$C$1^2+2*$C$1+4)/(($C$1^2+2*$C$1+2)^2)-((B4^2+3*B4+1)/(B4^2+2*B4+2)-(-B4^2+3*(-B4)+1)/(-B4^2+2*(-B4)+2))/(2*B4)

9.2. Select cell range A3:A103, C3:D103.

9.3. Command INSERT/CHARTS and select the type of a diagram *Scatter with straight lines*. As a result, we will obtain the diagram as shown in the picture below:



Example 4.2. Find the first derivative of the function $y = 4x^2 - 4x + 6$ at the point $x = 2$. Note that the derivative of the function at the point $x = 2$ equals 12. It was calculated by analytical method. We need the value of the derivative in order to compare it with the result we obtained by numerical calculation in a spreadsheet.

From the above it is known that the expression for calculation the derivative of the function of one variable in the point x has the following form:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h},$$

where h — is small finite number. It means that it is possible to take a small number instead of h. For example, we can take 0,00001 instead of h.

Remark. The number of dots after comma for h depends on which accuracy for a calculation of derivative is needed. For example, if we need to calculate the derivative with accuracy of two points after comma, it is enough to take h equal to 0,0001.

Solution.

It is possible to solve the task in two ways.

Approach 1.

1. Enter argument value, which is equal to 2 into C2. Enter a small increment of the argument in the cell C3. For example, this value can be 0.00001. In the cell C4 we calculate the sum C3=C1+C2.
2. Enter the following formula for calculating a derivative into the cell:

=((4*C4^2-4*C4+6)-(4*C2^2-4*C2+6))/C3.

3. After pressing Enter we obtain the result of calculation equal to 12,00004 (see the figure).

	A	B	C	D	E
1					
2		x	2		derivative
3		h	0,00001		12,00004
4		x+h	2,00001		

Approach 2.

1. Define the neighborhood of the point $x = 2$ of a small size. For example, the value from the left X_k =1,99999 and the value from the right X_{k+1} =2,00001. Next, enter these values into the cells B3 and B4 correspondingly.
2. Enter the formula of the right side of the given functional dependence into the cell C3 as shown in the picture. Make reference to the cell B3 where the value of x is written:

= 4*B3^2 -4*B3+6.

3. Copy this formula into the cell C4.
4. Enter the formula for calculating derivative into the cell E3:

=(C4-C3)/(B4-B3).

As a result, we obtain an approximate value of a derivative of the given function in the cell E3 in the point equal $x = 2$.

The value is equal to 12, which corresponds to the result obtained by analytical approach.

	A	B	C	D	E
1					
2		x	2		derivative
3		1,99999	13,99988		
4		2,00001	14,00012		12

Example 4.3. Find the first derivative of the function $y = \sin^2\left(1 + \frac{1}{x}\right)$ in the point $x = \frac{3\pi}{14}$.

From the above it is known that the expression for calculating the first derivative of a function of one variable in the point x has the following form:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h},$$

where h — is very small real number. It means it is possible to take small number instead of h. For example, we can take 0,00001 instead of h.

Solution.

1. Enter the given argument "3*pi/14" into the cell C2. Enter small argument increment into the cell C3. For example, it can be 0,00001. Calculate the sum C3=C1+C2 in the cell C4.
2. Enter the following formula for calculating derivative into the cell E3:

=((SIN(1+1/C4))^2-(SIN(1+1/C2))^2).

3. After pressing Enter we will obtain an value of a derivative which is equal to 0,000021

	A	B	C	D	E
1					
2		x	0,673198		derivative
3		Δx	0.00001		0,00002133215415
4		x+Δx	0,673208		

Example 4.4. Find the second derivative of the function $y = 4x^3 - 2x^2$ in the point $x = 2$. Note that the second derivative of the reduced function in the point $x = 2$ equals 40. It was

calculated by analytical approach. This is the value we will need to verify the result that was obtained by calculating the numerical method in the spreadsheet.

It is known from mathematics that expression for calculating second derivative of a function of one variable on the point x has the following form:

$$f''(x_0) \approx f_0^{(2)}(x_0,h) = \frac{1}{h^2}\big(f(x_0+h) - 2f(x_0) + f(x_0-h)\big).$$

where h — is small finite number. It means we can take small number instead of h. For example, we can take 0,00001 instead of h.

Solution.

1. Enter given argument value "2" into the cell C2. Enter small increment into the cell C3. For example, it can be 0,00001. Calculate the sum C3=C1+C2 in the cell C4.
2. Enter the following formula for calculating the second derivative into the cell E3:

=(1/(C3^2))*(4*C4^3-2*C4^2-2*4*C2^3+2*2*C2^2+
4*(C2-C3)^3-2*(C2-C3)^2).

3. After pressing *Enter* we obtain 44,00003917 (see the figure).

	A	B	C	D	E
1					
2		x	2		derivative
3		h	0,00001		44,00003917
4		x+h	2,00001		

Derivative of a function defined by a table

In case of a function defined by table we have: a discrete set of an argument's value (x_i) is matched by the set of values of function (y_i), $y_i = f(x_i)$, $i = 0,1,2,\ldots,n$. The increment is a difference between neighboring argument values, $\Delta x = h$, constant.

The derivative of a function y_1' in a knot $x = x_1$ can be found using finite differences in several ways:

✓ Left differences: $\Delta y_1 = y_1 - y_0$, $\Delta x = x_1 - x_0 = h$, $y_1' \approx \frac{y_1 - y_0}{h}$;

✓ Right differences: $\Delta y_1 = y_2 - y_1$, $\Delta x = x_2 - x_1 = h$, $y_1' \approx \frac{y_2 - y_1}{h}$;

✓ Central differences: $\Delta y_1 = y_2 - y_0$, $\Delta x = x_2 - x_0 = 2h$, $y_1' \approx \frac{y_2 - y_0}{2h}$.

Note that the formula for calculating the second derivative is:

$$y_1'' \approx \frac{y_2 - 2y_1 + y_0}{h^2}.$$

Example 4.5. Find the derivative of a function defined by the table. The function $y = 6x^4 - 4x^2$, where $x = 0{,}5;1{,}5;2{,}5\ldots.$. Find the derivative in the point $x_2 = 2{,}5$.

The formula for the derivative of the function defined by the table:

$$y_1' \approx \frac{y_2 - y_0}{2h}.$$

Solution.

1. Enter argument value x_1 equal to 1,5 into the cell C3. Enter argument value x_2 equal to 2,5 into the cell C4. Enter argument value x_3 equal to 3,5 into the cell C5. Find $2h$ by formula C6=C5-C3 in the cell C6.
2. Enter the following formula into the cell D3:

=6*C3^4-4*C3^2.

3. Copy this formula into the cell D5.
4. Enter the following formula for calculating derivative of the function in the point x_2 into the cell: (D5-D3)/C6.
5. After pressing *Enter* the result is 415,00 (see the figure).

	A	B	C	D	E
1					
2			x	y	
3		x1	1,5	21,375	derivative
4		x2	2,5	209,375	415
5		x3	3,5	851,375	
6		2h	2	80	

Exercises for independent work

1. Find the first derivative of $y = 3\cos^3(x)$ in the point $x = \frac{\pi}{2}$.

2. Find the first derivative of $y = 2\lg^2(x)$ in the point $x = 10$. Get the result in two ways.

3. Find the first derivative of the function $y = \ln^3(x) + 3x^2 - \log_2(x)$ in the point $x = 8$. Get the result in two ways.

4. Find the second derivative of $y = 3\cos^3(x)$ in the point $x = \frac{\pi}{2}$.

5. Find the second derivative of $y = 2\lg^2(x)$ in the point $x = 10$.

6. Find the second derivative of $y = \ln^3(x) + 3x^2 - \log_2(x)$ in the point $x = 8$.

7. Find the derivative of a function defined by table. Function $y = 3x^2 + 2x^3$, where $x = 1,2,3,\ldots..$ Find the derivative in the point $x_3 = 4$.

8. Find the derivative of a function defined by table. Function $y = 2^x$, where $x = 1,2,3,\ldots..$ Find the derivative in the point $x_2 = 3$.

5. Tangent line

Introduction

Let's refer to the problem, related with finding of the tangent to the graph of the function $f(x)$ in the point x_0:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Let's consider the function $f(x) = xe^{-x}$ in the point $x_0 = 0,5$ as an example. It is not hard to get the exact (analytical) finding equations of tangent here:

$$f(x_0) = f(0,5) = 0,5e^{-0,5}$$

$$f'(x) = \left(xe^{-x}\right)' = (x)' e^{-x} + x\left(e^{-x}\right)' = e^{-x} + xe^{-x}(-1)$$

$$f'(x) = e^{-x}(1 - x)$$

$$f'(x_0) = f'(0,5) = e^{-0,5}(1 - 0,5)$$

In the end, the equation of the required line takes the form:

$$y = 0,5e^{-0,5} + e^{-0,5}(1 - 0,5)(x - 0,5).$$

Then, after opening the bracket, we, finally, get:

$$y = 0,3 + 0,3(x - 0,5) = 0,3x + 0,15.$$

Let's use MS Excel for the visualization of the meaning of the tangent.

Example 5.1. Building a tangent line to the graph of the function with the use of the exact value of the derivative.

Let's choose some neighborhood of the point $x_0 = 0,5$, for instance, $x \in [-1;4]$. Enter the corresponding values into the cells of the MS Excel worksheet. Find points on the graphs of the function and its tangent line, using the functions mentioned above, then call the scatter chart with smooth curves and after pointing out the necessary references to columns of data and get the desired form (look at the fig. 5.1).

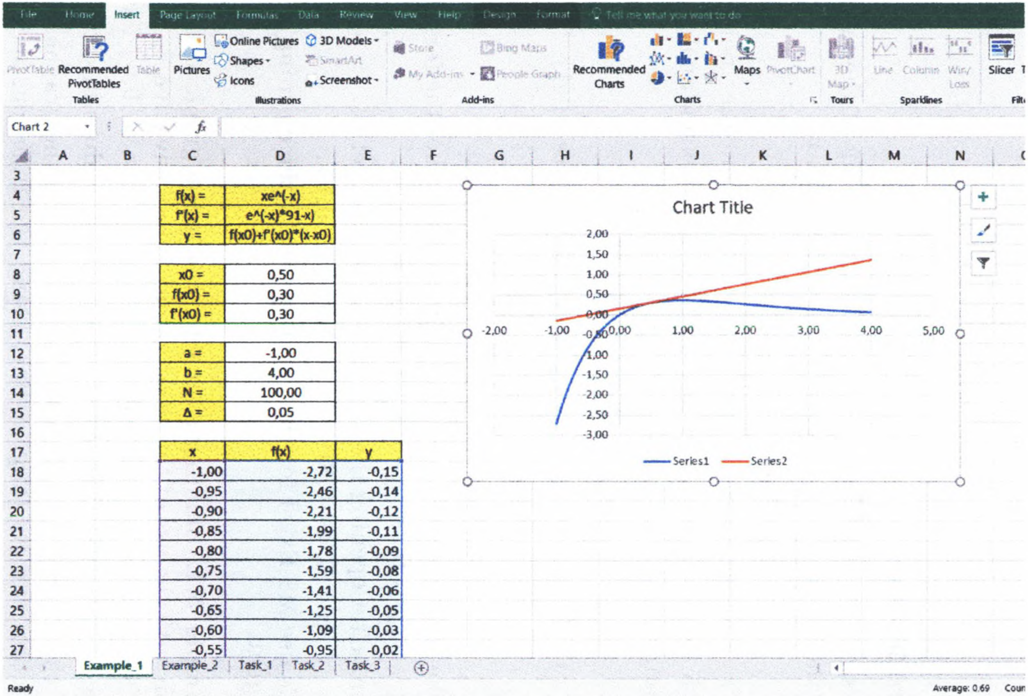


Fig. 5.1.

Remark. To expand the image, roll the wheel of the computer mouse, pressing Ctrl.

As usual, cells with text explanation, which cannot be calculated in MS Excel but only serve to clarify further programmed formulas, are highlighted by yellow color. In the first group of such “explaining” highlighted cells C4:D6 we indicated the function, its accurately calculated derivative and the formula for calculation of the ordinate y of the tangent for the graph:

	C	D
4	f(x) =	xe^(-x)
5	f'(x) =	e^(-x)*(1-x)
6	y =	f(xo)+f'(xo)*(x-xo)

We advise you to comment your calculations and color such cells for more comfortable further usage.

In the next group of cells C8:D10 we indicate the value of touch point x_0 and now we can calculate the value of the function $f(x_0)$ and its derivative $f'(x_0)$ according to the formulas mentioned above:

	C	D
8	xo =	0,50
9	f(xo) =	0,30
10	f'(xo) =	0,30

or in the “software” form:

	C	D
8	xo =	0,50
9	f(xo) =	=D8*EXP(-D8)
10	f'(xo) =	=(1-D8)*EXP(-D8)

(Absolutely accidentally the values of $f(0{,}5)$ and $f'(0{,}5)$ in this example match, even though we calculated them using different formulas.)

Next we indicate parameters of display of graphs: the beginning and the end of the considered range $[a,b]$ of changings of x, quantity of N splitting of the segment to generate points of the graph of the function and the resulting increment of the argument $\Delta = \dfrac{b - a}{N}$:

	C	D
12	a =	-1,00
13	b =	4,00
14	N =	100
15	Δ =	0,05

or in the "software form":

	C	D
12	a =	-1,00
13	b =	4,00
14	N =	100
15	Δ =	=(D13-D12)/D14

Now we should generate the sequence of values x, which runs through all the segment from a to b in increments Δ and calculate them in the resulting points of function and tangent value (cells C18:E118):

	C	D	E
	x	f(x)	y
18	-1,00	-2,72	-0,15
19	-0,95	-2,46	-0,14
...	...	...	...
118	4,00	0,07	1,36

Here we should pay attention to the "programming" of cells x. The first value in the column x, of course, becomes as a reference to the beginning of the segment a, but next ones are programmed as a reference to the previous value x «plus» a step Δ:

	C	D	E
	x	f(x)	y
18	=D12	-2,72	-0,15
19	=C18+$D$15	-2,46	-0,14
20	=C19+$D$15	-2,21	-0,12
...	....	...	...

Do not forget to press button *F4* freezing values of Δ after the link to the cell D15 (a step Δ) to further copy of the formula to be correct. Remember that for this we need to drag the programmed cell downwards by the left mouse button pressed on the special cross which will appear in the right bottom corner of the cell after hovering over it with the mouse cursor.

Other cells of values of $f(x)$ and y are calculated by standard way according to the formulas mentioned above:

	C	D	E
	x	f(x)	y
18	=D12	=C18*EXP(-C18)	=$D$9+$D$10*(C18-$D$8)
19	=C18+$D$15	=C19*EXP(-C19)	=$D$9+$D$10*(C19-$D$8)
...	....	...	...

And also do not forget to fixate the values in cells $f(x_0)$ and $f'(x_0)$ during the calculations of the ordinates of the tangent y.

The last step is to create the corresponding diagram to plot the function and the tangent. For that we should highlight the entire numeric range of the last three columns with data with the left mouse button and choose in *Insert* $\rightarrow$ *Charts* $\rightarrow$ *The scatter charts* $\rightarrow$ *The scatter charts with smooth curves* (look at the fig. 5.2).

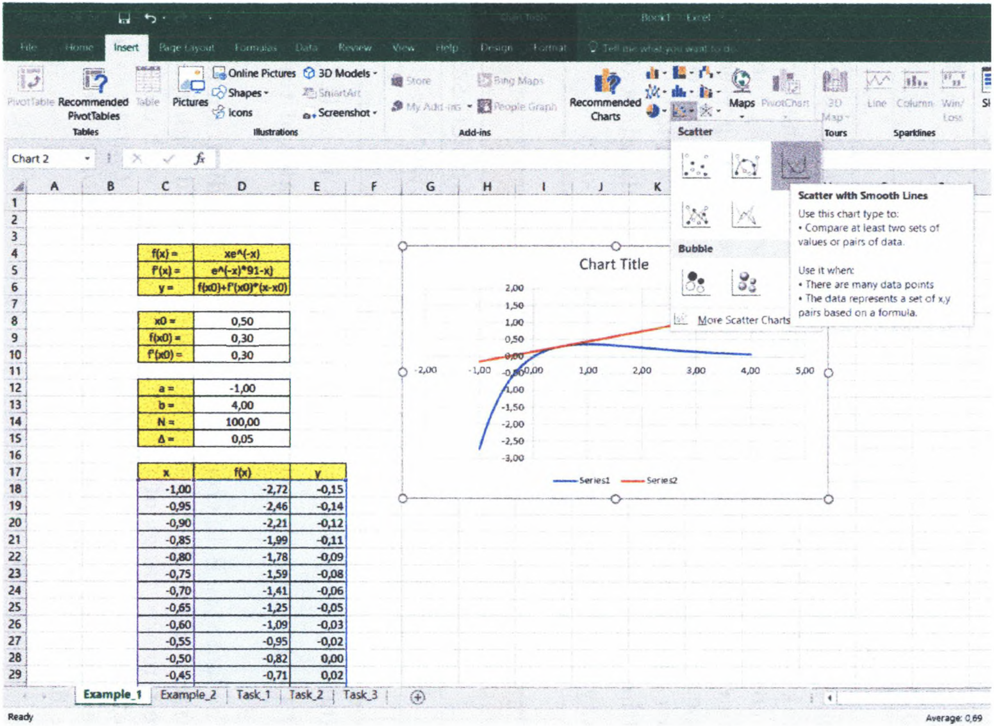

File
Home
Insert
Page Layout
Formulas
Data
Review
View
Help
Design
Format
PivotTable
Recommended PivotTables
Table
Tables
Pictures
Online Pictures
Shapes
Icons
3D Models
SmartArt
Screenshot
Illustrations
Store
My Add-ins
Bing Maps
People Graph
Add-ins
Recommended Charts
Maps
PivotChart
3D Map
Tours
Line
Column
Win/Loss
Sparklines
Scatter
Bubble
More Scatter Charts
Scatter with Smooth Lines
Use this chart type to:
• Compare at least two sets of values or pairs of data.
Use it when:
• There are many data points
• The data represents a set of x,y pairs based on a formula.
Chart 2
f(x) =
xe^(-x)
f'(x) =
e^(-x)*91-x)
y =
f(x0)+f'(x0)*(x-x0)
x0 =
0,50
f(x0) =
0,30
f'(x0) =
0,30
a =
-1,00
b =
4,00
N =
100,00
Δ =
0,05
x
f(x)
y
-1,00 -2,72 -0,15
-0,95 -2,46 -0,14
-0,90 -2,21 -0,12
-0,85 -1,99 -0,11
-0,80 -1,78 -0,09
-0,75 -1,59 -0,08
-0,70 -1,41 -0,06
-0,65 -1,25 -0,05
-0,60 -1,09 -0,03
-0,55 -0,95 -0,02
-0,50 -0,82 0,00
-0,45 -0,71 0,02
Chart Title
Series1
Series2
Example_1
Example_2
Task_1
Task_2
Task_3
Ready
Average: 0,69

Fig. 5.2.

We are close to the end. Changing callings of the inscriptions is left. It is done in the following way: highlight the diagram with graphs, pointing out free (empty, free from the graphs) place on the diagram, then click the right mouse button and press *"Select data..."* (look at the fig. 5.3). If you miss and point out not the free place of the diagram but, for instance, the grid line or label area of the chart legend, then other options menu will appear. Also pay attention to the fact that after pressing the right upper cross of the diagram we get into the menu of the visual settings of the diagram, where it is mentioned: whether to display the text explanations.

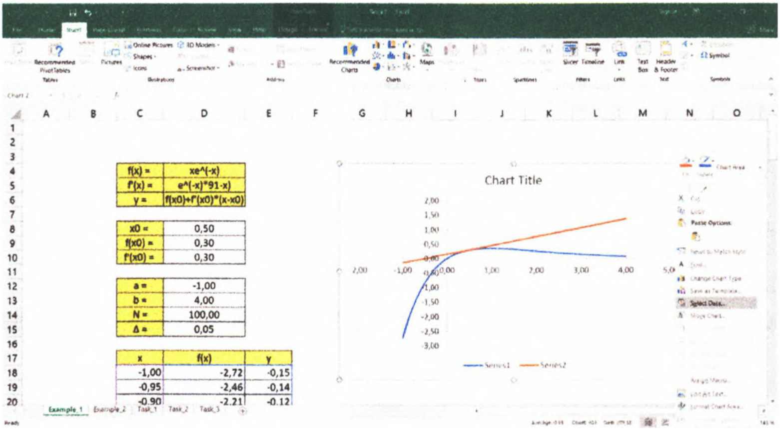


Fig. 5.3.

After calling the *"Select Data..."* the dialog window opens where we can choose to modify, enter or delete various data (look at Fig. 5.4)

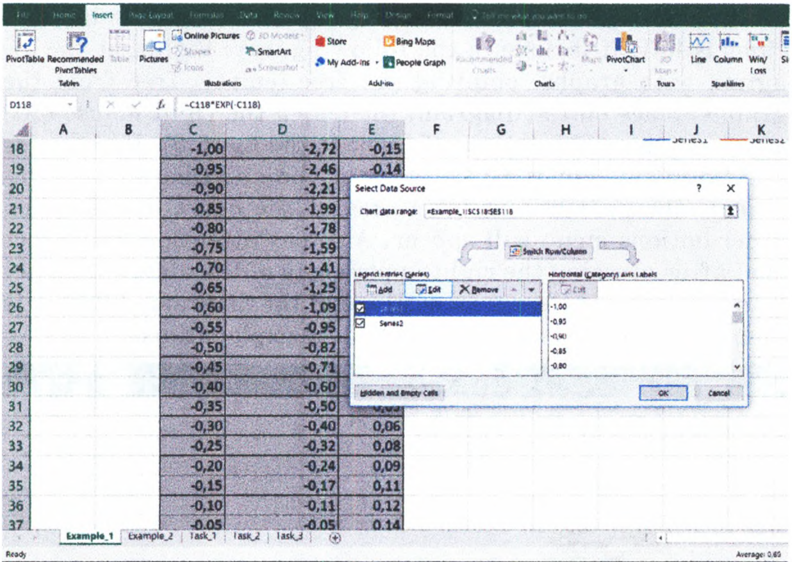


Fig. 5.4.

and edit the data ranges and their labels (look at the fig. 5.5)

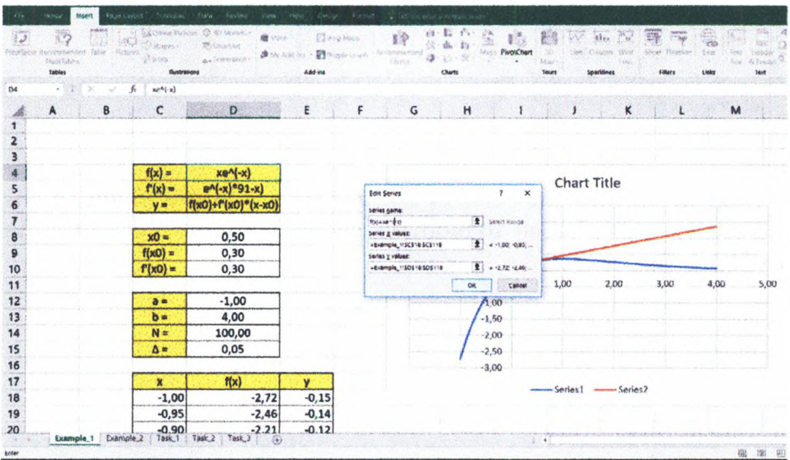


Fig. 5.5.

Example 5.2. Construction of a tangent line to the graph of the function using approximate values of its derivative.

There are such cases, when the accurate calculation of the derivative is impossible or cannot be done from a computational point of view. The cases of the first type are those when the value of the function is not done analytically and the information about the values of the function is available till the fixed value of the argument (often till the fixed moment of time), for example, when there are new processes, which are evaluating at current time. The cases of the second type of the irrelevance of the analytical calculations of the derivative are those huge difficult functions, which we need to program tediously to fight for the absolutely tiny increase of accuracy.

We see the exact repeat of the previous argumentations which only differ in the calculation of the derivative of the function $f'(x)$ (look at the fig. 5.6)

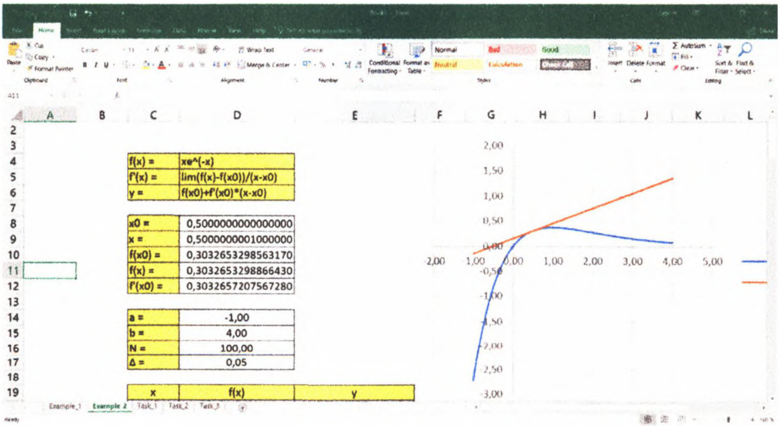


Fig. 5.6.

There is no accurate (analytical) calculation of the derivative. Instead of this, the definition of the derivative is mentioned:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

Which is changed to the approximate equality if the values of x and x_0 are really closed

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \approx \frac{f(x) - f(x_0)}{x - x_0}$$

	C	D
4	f(x) =	xe^(-x)
5	f'(x) ~	(f(x)-f(xo))/(x-xo)
6	y =	f(xo)+f'(xo)*(x-xo)

Thus we need to give not only the value of the x_0 but also close to it the value of the x. For example:

$$x = x_0 + 0,\underbrace{0000000001}_{10\text{ decimals}}.$$

After the re-programming the calculation of the derivative of the function we get:

	C	D
8	xo =	0,5000000000000000
9	x =	0,5000000001000000
10	f(xo) =	0,3032653298563170
11	f(x) =	0,3032653298866430
12	f'(xo) =	0,3032657207567280

Alternatively, in MS Excel code:

	C	D
8	xo =	0,5000000000000000
9	x =	=D8+0,0000000001
10	f(xo) =	=D8*EXP(-D8)
11	f(x) =	=D9*EXP(-D9)
12	f'(xo) =	=(D11-D10)/(D9-D8)

We can compare with the exact value of the derivative from the previous task, mentioning the same precision to 16 characters in the format of the cells (click on the right button of them in the item "the format cells"):

Exact $f'(x_0) = 0{,}3032653298563170$

Approximate $f'(x_0) = 0{,}3032657207567280$

(the difference is about four tenmillionths).

Then it all goes as in the previous example. In the end we get the same picture (look at the fig.5.7).

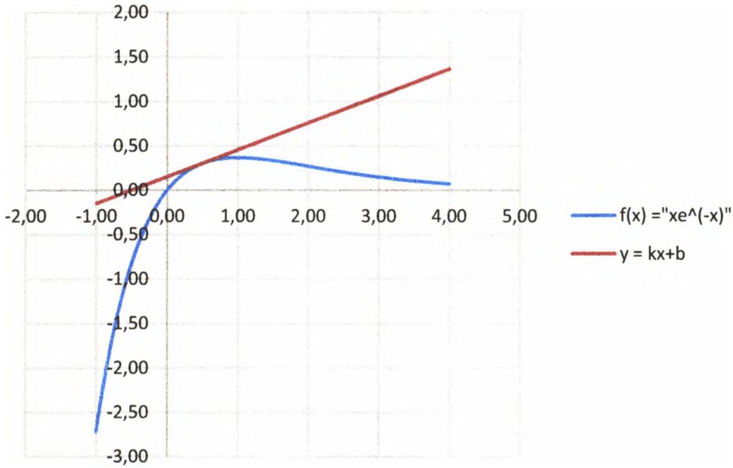


Fig. 5.7.

Remark. In the cases when it is necessary to calculate the close value of the derivative of the function for which next values are unknown (it is about the forecast), the calculations of x close to x_0 are held according to the next formula:

$$x = x_0 - 0{,}\underbrace{0000000001}_{\text{10 decimals}}.$$

Also, it should be mentioned that during the usage of the close values of the derivative it is necessary to look at the smoothness and regularity of the function. If the function is irregular and/or singular (explosive, abrupt, unlimited and etc.), the derivative should be calculated analytically.

Exercises for independent work

1. Depict the function $x^3 - 3x^2 + x - 8$ with its tangent at the value 1,5. Instruction: the derivative should be calculated analytically exactly, take the segment $[-1;4]$ as the neighborhood of the point 1,5 .

2. Depict the function x^x with its tangent at the value 2. Instruction: the derivative should be calculated closely, take the segment $[0,1;3]$ as the neighborhood of the point 2.

3. * How much times does the tangent to the function $x^2 + 8\sin x - 10$, held in point 2, crosses the whole graph?

4. ** Compare the results of the previous exercises №1–3 using the exact and close values of derivatives of the functions.

6. Local Extrema

Monotonicity and the searching for local extrema of a function

Introduction

Increasing, non-increasing, decreasing and non-decreasing functions on the set D are called *monotonic* on this set, and increasing and diminishing functions are *strictly monotonic.* The intervals, in which the function is monotonic, are called *monotonicity intervals.*

Necessary conditions for increasing (decreasing) function. If the differentiated function $f(x)$ on the interval $(a;b)$ increases (decreases), it will be $f'(x) \geq 0$ $(f'(x) \leq 0)$ for each $x \in (a;b)$.

Sufficient conditions for increasing (decreasing) function. If the function $f(x)$ is differentiable on the interval $(a;b)$ and $f'(x) > 0$ $(f'(x) < 0)$ for any $x \in (a;b)$, then the function increases (decreases) on the interval $(a;b)$

Example 6.1. Find intervals of monotonicity of $f(x) = x^3 - 3x - 4$.

Solution. The function is defined on the entire real axis: $x \in (-\infty;+\infty)$.

Let's find the derivative of the function: $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ $f'(x) > 0$ *and* $x \in (-\infty;-1) \cup (1;+\infty)$ *and* $f'(x) < 0$ *at* $x \in (-1;1)$

The answer: The function increases on the intervals $(-\infty;-1) \cup (1;+\infty)$ and decreases in the interval $(-1;1)$.

The point x_0 from the domain of the function $f(x)$ is called minimum (maximum) point of this function, if there is such a δ-neighborhood $(x_0 - \delta; x_0 + \delta)$ of the point x_0, which is for all $x \neq x_0$ in this neighborhood the inequality $f(x) > f(x_0)$ $(f(x) < f(x_0))$ is satisfied.

The minimum and maximum points are called *extrema points* and the function values at these points are called *extrema of the function.*

The concept of extremum is always connected with a certain neighborhood of the point from the domain of the function. That's why the function can have got extremum only at internal points of the domain. If the function has several extremums, they are called *local (relative).*

The necessary condition for extremum. If the differentiable function $y = f(x)$ has an extremum at the point x_0 then its derivative at this point equals to 0: $f'(x_0) = 0$.

Geometrically, the equation $f'(x_0) = 0$ means that at the extremum point of the differentiable function $y = f(x)$ the tangent to its graph is parallel to the Ox axis.

However, the converse is not true, i.e. if $f'(x_0) = 0$, it doesn't mean that x_0 — an extremum point. For example, for the function $y = x^3$ its derivative is: $y' = 3x^2$ and equals to zero at $x = 0$, but $x = 0$ isn't the extremum point (this can be seen from the graph of the function).

There are functions that don't have a derivative at extremum points. For example, the continuous function $y = |x|$ at the point $x = 0$ doesn't have a derivative, but the point $x = 0$ is the minimum point (you can also see it from the graph of the function).

Thus, a continuous function can have an extremum only at points where the derivative of the function is zero or does not exist. Such points are called *critical points.*

A sufficient condition of the extremum. If the continuous function $y = f(x)$ is differentiable in some δ-neighborhood of the critical point x_0 and when passing through it (from left to right) the derivative $f'(x)$ changes the sign from plus to minus, then x_0 is the maximum point, if from minus to plus, then x_0 is the minimum point.

To study the function at an extremum means to find all its extrema.

The rule for the study of the function at extremum:

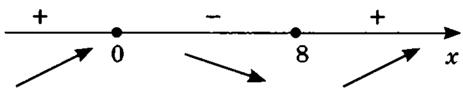
1. Find critical points of the function $y = f(x)$;
2. Choose from them only those which are internal points of the domain of the function;
3. Explore the sign of the derivative $f'(x)$ to the left and right of each of the selected critical points;
4. In accordance to the sufficient condition for the extremum, write out the extremum points (if any) and calculate the values of the function in them.

Example 6.2. Find the extremum of the function $y = \frac{x}{3} - \sqrt[3]{x^2}$.

Solution. Obviously, $D(y) = R$ (i.e. entire real line).

We find the derivative of the function $y' = \frac{1}{3} - \frac{2}{3 \cdot \sqrt[3]{x}} = \frac{1}{3} \cdot \frac{\sqrt[3]{x} - 2}{\sqrt[3]{x}}$.

The derivative does not exist at $x_1 = 0$ and equals to zero at $x_2 = 8$. These points divide the entire domain into three intervals $(-\infty;0),(0;8),(8;+\infty)$. Note the signs of the derivative on the left and right of each one of the critical points:



Therefore, $x_1 = 0$ is the maximum point, $y_{\max} = y(0) = 0$ and $x_2 = 8$ is the minimum point, $y_{\min} = y(8) = -\frac{4}{3}$.

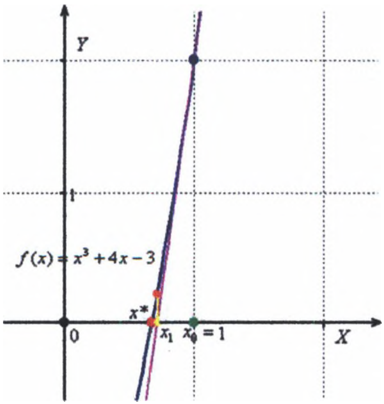
Sometimes it is convenient to use another sufficient condition of the existence of an extremum, based on the determination of the sign of the second derivative.

The second sufficient condition for extremum. If at the point x_0 the first derivative of the function $f(x)$ is equal to zero $f'(x_0) = 0$, and the second derivative at the point x_0 exists and is different from zero ($f''(x_0) \neq 0$), then for $f''(x_0) < 0$ at the point x_0, the function has a maximum and a minimum at $f''(x_0) > 0$.

One of the more effective ways to find the approximate value of the root is the *tangents method*.

Example 6.3. Using the graphical method, find the interval $[a;b]$ on which the actual root x^* of the function $x^3 + 4x - 3 = 0$ is located. Using the Newton's method, obtain an approximate value of the root with accuracy 0,001.

The short geometric essence of the method consists in the following: firstly, choose one end of a segment with the help of a special criterion. This end is called the initial root approximation, in our example: $x_0 = 1$. Now draw a tangent to the graph of the function $f(x) = x^3 + 4x - 3$ at the point with abscissa $x_0 = 1$ (blue dot and violet tangent):



This tangent crossed the X-axis at the yellow point, and note that at the first step we have almost "hit the root"! This will be the first approximation of the root x_1. Further, construct the yellow perpendicular to the function graph and "get" the orange dot. Through the orange dot, we again draw a tangent that will cross the axis even closer to the root! And so on. It is not difficult to understand that, using the tangents method, we are approaching the goal quite quickly, and it will take literally several iterations to achieve an accuracy of $\varepsilon = 0{,}001$.

Solution. The first step is to reveal the root graphically. This can be done by plotting the function $f(x) = x^3 + 4x - 3$.

Thus, the desired root belongs to the interval $[0;1]$ and is approximately equal to 0,65.

At the second step, select the initial approximation x_0 of the root. This is usually one of the segment ends. The initial approximation must satisfy the following condition: $f(x_0) \cdot f''(x_0) > 0$.

Let's find the first and second derivatives of the function $f(x) = x^3 + 4x - 3$

$f'(x) = (x^3 + 4x - 3)' = 3x^2 + 4, \quad f''(x) = (3x^2 + 4)' = 6x.$
As the initial approximation, we choose $x_0 = 1$.

At the third step, each subsequent approximation of the root x_{n+1} is calculated on the basis of previous data using the following recurrent formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The process terminates when the condition $\left|\frac{f(x_n)}{f'(x_n)}\right| < \varepsilon$ is met, where ε is the predetermined accuracy of the calculations. As a result, the approximate value of the root is taken as the "n" approximation: $x^* \approx x_n$.

In practice, it is convenient to enter the results of calculations into a table, while in order to somewhat shorten the record, the fraction is often denoted by $h_n = \dfrac{f(x_n)}{f'(x_n)}$

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = \dfrac{f(x_n)}{f'(x_n)}$
0	1	2	7	0,28571
1	0,71429	0,22157	5,53061	0,04006
2	0,67422	0.00338	5,36373	0,00063

The calculations are carried out in MS Excel — it is much more convenient and faster.

Determine the areas of increasing and decreasing functions. Study the function at local extremum. Critical points.

Example 6.4.

Find the points of local extrema and the areas of increasing and decreasing of the function

$$y = x \cdot e^{-x^2}.$$

Solution.

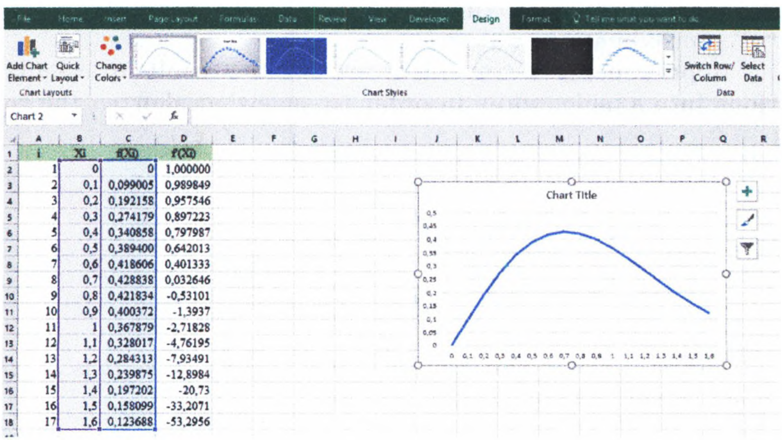
Derivative of the function:

$$y' = e^{-x^2} - 2x^2 e^{-x^2} = \left(1 - 2x^2\right)e^{-x^2}.$$

$$y' = 0 \ \ at \ \ x = \pm\frac{\sqrt{2}}{2} \approx \pm 0{,}707.$$

Construct a graph of the function and its derivative on an Excel sheet:

1. Fill A1 to D1 as follows — i, Xi, f(Xi), f′(Xi).
2. Fill the A2-A18 — values from 1 to 17
3. Put in the sell B2 the value 0
4. Then fill the sell B3 with =B2+0.1 and drop till B18
5. Write down =$B2*EXP($B2*$B2*-1) into C2 and drop it to the C18
6. Draw a graph by selecting sells B2:C18 then tap the *Insert* bottom and then press the *Insert Line*.



i	Xi	f(Xi)	f′(Xi)
1	0	0	1,000000
2	0,1	0,099005	0,989849
3	0,2	0,192158	0,957546
4	0,3	0,274179	0,897223
5	0,4	0,340858	0,797987
6	0,5	0,389400	0,642013
7	0,6	0,418606	0,401333
8	0,7	0,428838	0,032646
9	0,8	0,421834	-0,53101
10	0,9	0,400372	-1,3937
11	1	0,367879	-2,71828
12	1,1	0,328017	-4,76195
13	1,2	0,284313	-7,93491
14	1,3	0,239875	-12,8984
15	1,4	0,197202	-20,73
16	1,5	0,158099	-33,2071
17	1,6	0,123688	-53,2956

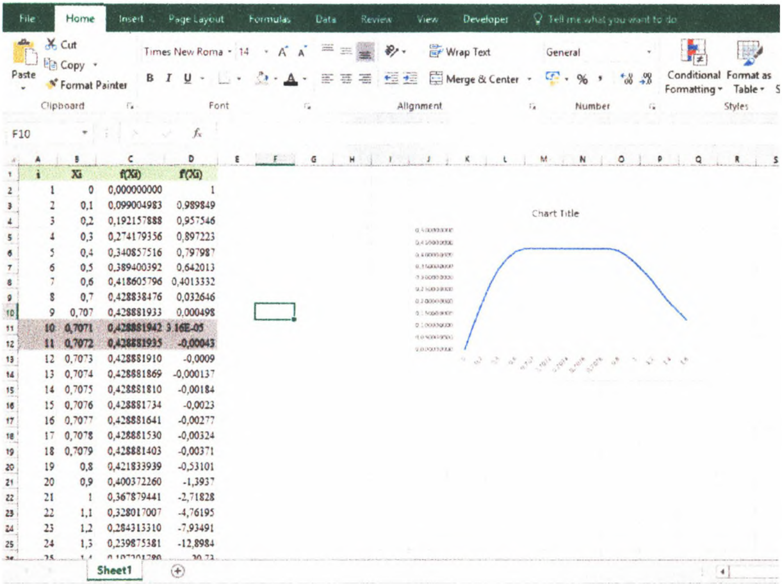
Fig. 6.1.1. Constructing the graph.

From the table it's seen that $x = \dfrac{\sqrt{2}}{2} \approx 0{,}707$ is the maximum point. As this function is odd, then $x = -\dfrac{\sqrt{2}}{2} \approx -0{,}707$ is the minimum point.

Therefore, the interval of increasing function $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, the intervals of decreasing function $\left(-\infty, -\dfrac{\sqrt{2}}{2}\right) \cup \left(\dfrac{\sqrt{2}}{2}, +\infty\right)$.

Below, we will consider two ways of more accurate determination of the values of the extremum points.

From the figure 6.1.1 one can see that the maximum point is between 0,7 and 0,8. Choose the range B10: D19 and move it down by 10 lines, then on the empty space in column B we set the values of x with step 0.0001 and calculate in columns C and D the values of the function and its derivative. Doing the same procedure number of times, we determine the values of the extremum point with the required accuracy.



i	Xi	f(Xi)	f'(Xi)
1	0	0,000000000	1
2	0,1	0,099004983	0,989849
3	0,2	0,192157888	0,957546
4	0,3	0,274179356	0,897223
5	0,4	0,340857516	0,797987
6	0,5	0,389400392	0,642013
7	0,6	0,418605796	0,4013332
8	0,7	0,428838476	0,032646
9	0,707	0,428881933	0,000498
10	0,7071	0,428881942	3,16E-05
11	0,7072	0,428881935	-0,00043
12	0,7073	0,428881910	-0,0009
13	0,7074	0,428881869	-0,000137
14	0,7075	0,428881810	-0,00184
15	0,7076	0,428881734	-0,0023
16	0,7077	0,428881641	-0,00277
17	0,7078	0,428881530	-0,00324
18	0,7079	0,428881403	-0,00371
19	0,8	0,421833939	-0,53101
20	0,9	0,400372260	-1,3937
21	1	0,367879441	-2,71828
22	1,1	0,328017007	-4,76195
23	1,2	0,284313310	-7,93491
24	1,3	0,239875381	-12,8984

Fig. 6.1.2. Finding the extrema points.

2nd method.

Using the Newton's method (the method of tangents) for solving equations allows us to solve this problem.

Xk	f(Xk)	F'(Xk)	Xk+1	dXk
0.7	0.012252528	-1.73251	0.707072136	0.00707214
0.707072136	5.94366E-05	-1.71561	0.70710678	3.4645E-05
0.70710678	1.45583E-09	-1.71553	0.707106781	8.4862E-10
0.707106781	-1.34677E-16	-1.71553	0.707106781	0

Fig. 6.1.3. Tangents method to determine the zero of derivative.

It is required to find the root of the derivative of a given function $f'(x) = 0$.

If $x = 0{,}7$ is the initial approximation, then, as a result of the iterative process, the following approximations are found by the formula:

$$x_{k+1} = x_k - f'(x_k) \,/\, f''(x_k).$$

The condition of the end of the process

$$|x_{k+1} - x_k| = 0$$

with computer precision.

In cell J34 there is the initial approximation x_k, in cell K34 is the calculated value of $f'(x_k)$, in cell L34 is the value of $f''(x_k)$, in cell M34 is the value of $x_{k+1} = x_k - f'(x_k) \,/\, f''(x_k)$, in cell N34 — the value $|x_{k+1} - x_k|$, in cell J35 the value x_{k+1} is copied for the next iteration. The process terminates by the condition: the value in cell N34 is 0. In this task

$$x_{k+1} = 0{,}707106781.$$

3rd method.

This task can be solved by using the MS Excel tool "Goal Seek".

To do this, follow these steps:

1. Place the initial approximate value (x_0) in the first cell.

2. In the second cell, place the formula that calculates the value of $f(x_0)$.

3. Call *DATA What-If Analysis Goal Seek*
4. Set the values of arguments as follows.
Set in cell: Address of the second cell
Value: 0
Changing the value of the cell: The address of the first cell.
Consider the use of this tool on the example of this task.
It is required to find the root of equation

$$\left(1 - 2x^2\right)e^{-x^2} = 0,$$

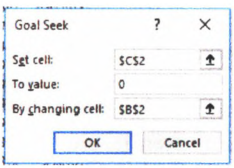
The initial approximation is x=0,7.

In cell B3 put the value 0,7; in the cell B32 calculate the appropriate value $\left(1 - 2x^2\right)e^{-x^2}$. Then turn to the *Goal Seek* procedure as it is shown above.

A	B	C	D
15	0,7076	0,428881734	-0,0023
16	0,7077	0,428881641	-0,00277
17	0,7078	0,428881530	-0,00324
18	0,7079	0,428881403	-0,00371
19	0,8	0,421833939	-0,53101
20	0,9	0,400372260	-1,3937
21	1	0,367879441	-2,71828
22	1,1	0,328017007	-4,76195
23	1,2	0,284313310	-7,93491
24	1,3	0,239875381	-12,8984
25	1,4	0,197201789	-20,73
26	1,5	0,158098837	-33,2071
27	1,6	0,123687585	-53,2956
0,7	0,007506222		

Fig. 6.1.4. "What-If Analysis" for determining the zero of derivative.

Set the arguments according to the following:



The result will be presented in the indicated cells as it is presented in the figure 6.1.5.

	A	B	C	D	E
23	22	1,1	0,328017007	-0,42344	
24	23	1,2	0,284313310	-0,445424	
25	24	1,3	0,239875381	-0,439156	
26	25	1,4	0,197201789	-0,411307	
27	26	1,5	0,158098837	-0,368897	
28	27	1,6	0,123687585	-0,318496	
29					
30					
31	0,707	0,0001832			

Fig. 6.1.5. The result of *"Goal Seek"*.

It should be mentioned that the procedure "Goal seek" allows us to calculate the selected value with the accuracy of 0,001. Therefore, if it's necessary to get the value with higher accuracy, it's better to use the Newton's method.

Example 6.5.

Find the local extrema and intervals of increasing and decreasing of the function $y = x\ln x$.

Solution.

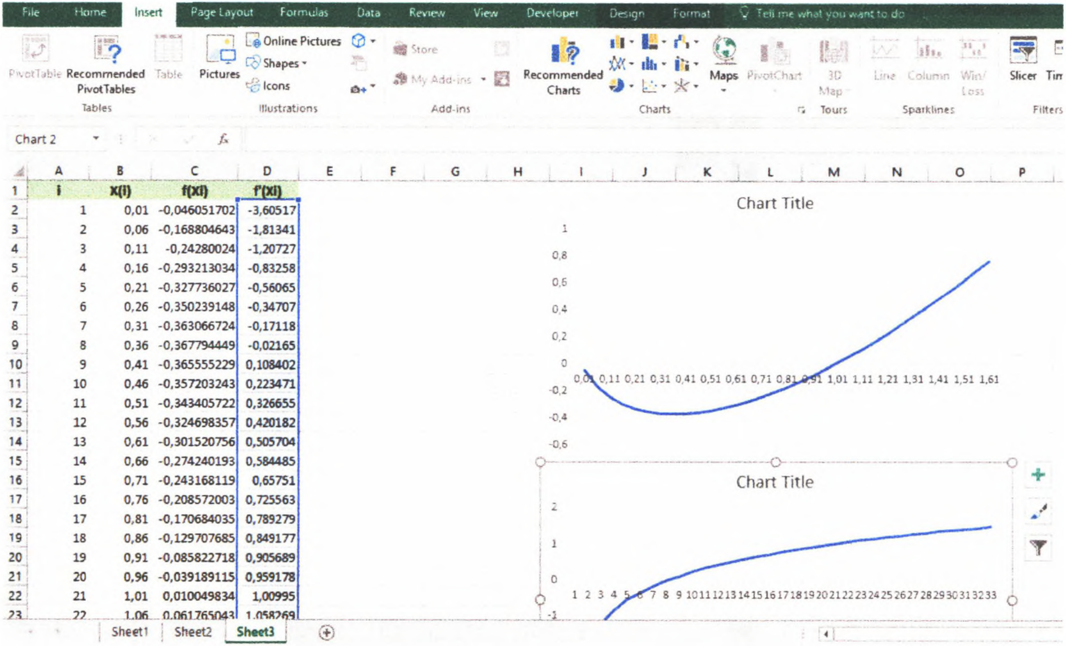
For the function $y = x\ln x$ the domain is $x > 0$. Calculate the table of values at $x = 0{,}01 + nh$, where $h = 0.05$, $n = 0,1,2,\ldots,32$.

1. Enter the numbers 1,2, ..., 33 into the range of cells A2:A34 of a MS Excel worksheet. In cell B2 enter the number 0,01. In the cell B3 enter the expression B2 + 0,05. Copy the formula to cell B34.

2. In cell C2 enter the formula = B2 * LN(B2). Copy the formula to the cell C34.

3. Select the range of cells B2:C34. Choose the INSERT/ CHARTS command, as shown in the figure.

Create the graph of the function and its derivative in the MS Excel worksheet.



i	x(i)	f(xi)	f'(xi)
1	0,01	-0,046051702	-3,60517
2	0,06	-0,168804643	-1,81341
3	0,11	-0,24280024	-1,20727
4	0,16	-0,293213034	-0,83258
5	0,21	-0,327736027	-0,56065
6	0,26	-0,350239148	-0,34707
7	0,31	-0,363066724	-0,17118
8	0,36	-0,367794449	-0,02165
9	0,41	-0,365555229	0,108402
10	0,46	-0,357203243	0,223471
11	0,51	-0,343405722	0,326655
12	0,56	-0,324698357	0,420182
13	0,61	-0,301520756	0,505704
14	0,66	-0,274240193	0,584485
15	0,71	-0,243168119	0,65751
16	0,76	-0,208572003	0,725563
17	0,81	-0,170684035	0,789279
18	0,86	-0,129707685	0,849177
19	0,91	-0,085822718	0,905689
20	0,96	-0,039189115	0,959178
21	1,01	0,010049834	1,00995
22	1,06	0,061765043	1,058269

Fig. 6.2.1. Constructing the graph of the function and its derivative

Therefore, the interval of increasing is $\left(\frac{1}{e};\ +\infty\right)$, and the interval of decreasing is $\left(0;\ \frac{1}{e}\right)$. The minimum point of the function is $x = 0{,}367879441$.

Example 6.6.

Find points of local extrema and the areas of increasing and decreasing of the function $y = \sqrt[3]{x^2}$.

Construct the function and its derivative in a new MS Excel worksheet.

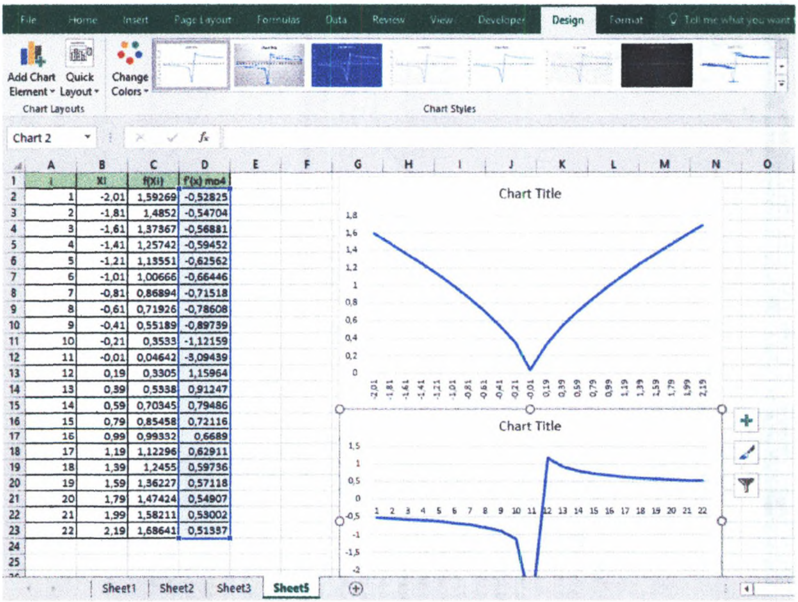


Fig. 6.3.1. Constructing a graph of the function and its derivative.

Example 6.7.

Study the function $f(x) = |\cos x|$ for local extrema. Find the points of local extrema and the areas of increasing and decreasing of the function $y = |\cos x|$.

Solution.

This function is even and periodic with a period equal to π. Therefore, it is sufficient to explore it on a segment of length equal to a half of the period.

Using the definition of the module, you can get another representation of the function:

$$f(x) = \begin{cases} \cos x, & \text{for } \cos x \geq 0, \\ -\cos x, & \text{for } \cos x < 0. \end{cases}$$

Therefore, the derivative of this function is determined as follows

$$f'(x) = \begin{cases} -\sin x, & \text{for } \cos x \geq 0, \\ \sin x, & \text{for } \cos x < 0, \end{cases}$$

or

$$f'(x) = -\operatorname{sgn}(\cos x) \cdot \sin x.$$

From the last formula it follows that the derivative $f'(x)$ turns to 0 at the values $\pi n,\ n \in N$. Moreover, at each of these points, the derivative changes the sign from plus to minus, which means that these points are the maximum points at which the function reaches values equal to 1. In addition, at points $\frac{\pi}{2} + \pi n, n \in N$, the derivative has points of the first kind of discontinuity (*jump*), at which its value changes from -1 to 1 (jumps in the derivative), thus, at these points the function reaches minima equal to 0. Using MS Excel, this problem can be solved as follows. In the MS Excel worksheet, we set the table of values of the function and its derivative on the interval [-1,8; 3,2], containing the period of this function. Let the values of the argument x increase with the step $\Delta = 0{,}02$. At the range B9: B259, we have the indicated values of the argument, in the C9:C259 range, calculate the corresponding function values using the formula ABS(COS(B9)), then continue to the other cells in the range, in the range D9: B259, in the range E9: E259 — the corresponding values of the function *sinx*, in the range of F9: F259, calculate the corresponding values of the derivative of the function using the formula IF (COS (B9) <0, SIN (B9), — SIN (B9)) . After that, we construct the graphs of the function and its derivative (see Figure 6.4.1).

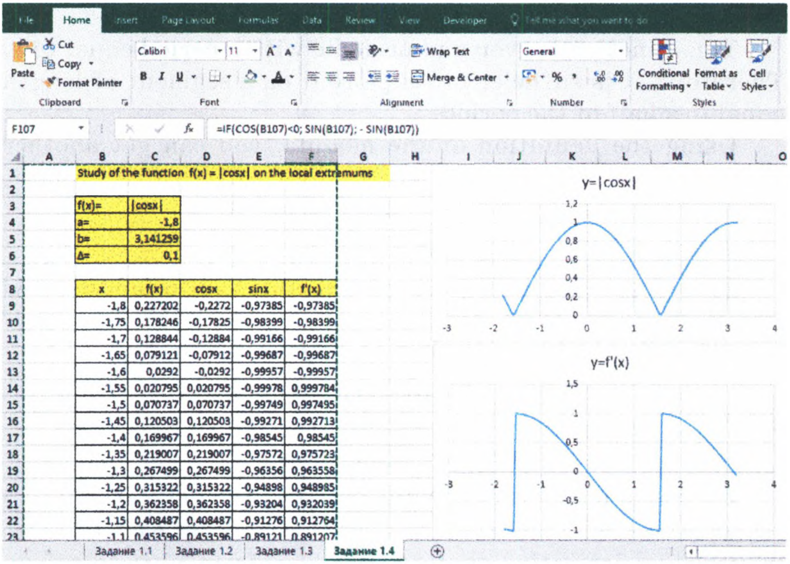


Fig. 6.4.1. Graphs of the function and its derivative

In Fig. 6.4.2. and Fig. 6.4.3 there are graphs of the function and its derivative. It is seen from the graphs that the function is continuous, but its derivative has points of discontinuity of the first kind (i.e. jump).

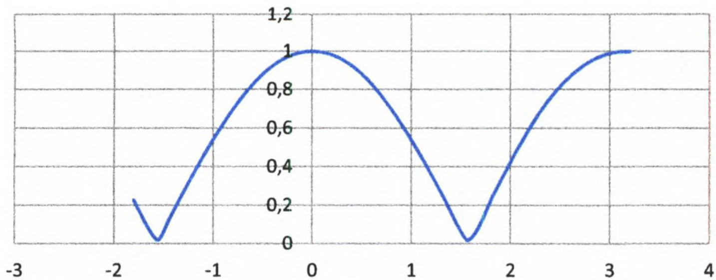


Fig. 6.4.2. The graph of the $f(x) = |\cos x|$.

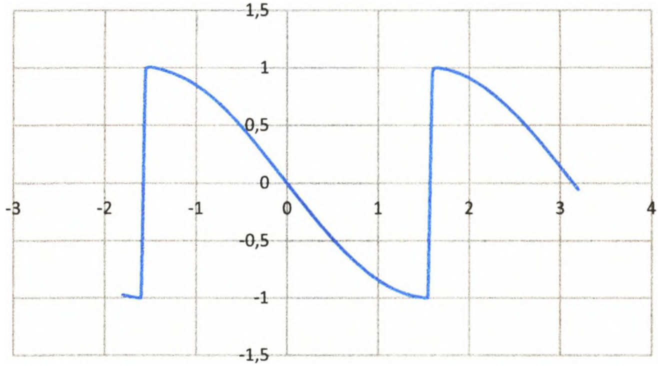


Fig. 6.4.3. The graph of the derivative $f'(x) = -\mathrm{sgn}(\cos x) \cdot \sin x$

Analyzing the graphs and taking into account the periodicity of the function, we conclude that at the points $x = \pi n$, $n \in N$, the function reaches its maximum values equal to 1, and at the points $\frac{\pi}{2} + \pi n$, $n \in N$ its minimum values equal to 0. Moreover, the points $\frac{\pi}{2} + \pi n$, $n \in N$ are the corner points of the graph of the function, since at these points the derivative of the function suffers a discontinuity. It should be noted that when you go through all these points, the derivative changes the sign.

Exercises for independent work

1. Find the points of local extrema and the intervals of increasing and decreasing of the function

$$y = \frac{(x+4)(x+2)(x-1)(x-5)}{x^4 + 16}.$$

2. Find the points of local extrema and the intervals of increasing and decreasing of the function $y = x\sqrt{4 - x^2}$.

3. Find the points of local extrema and the intervals of increasing and decreasing of the function $y = \ln\sqrt{1 + x^2}$.

7. Concavity

Numerical study of a convexity of a function and finding its points of inflection.

Introduction

The graph of a differentiable function $y = f(x)$ is called *concave upward* (*convex*, or *convex downward)* in the interval $(a;b)$, if it is located higher than any of its tangent lines on this interval. The curve chart $y = f(x)$ is called *concave downward* (*concave*, or *convex upward*) in the interval $(a;b)$, if it is located lower than any of its tangent lines in this interval.

The point of the graph of the continuous function $y = f(x)$, separating its parts of different convexity, is called the *inflection point.*

The condition of a concavity in the interval. If the function $y = f(x)$ at all of the points of the interval $(a;b)$ has a negative second derivative, that is, $f''(x) < 0$, then the graph of the function on that interval is convex upward. If $f''(x) > 0$ at all of the points of the interval $(a;b)$, then the graph is convex downward.

A sufficient condition for the existence of inflection points. If the second derivative $f''(x)$ changes its sign, when passing through the point x_0, at which it is equal to zero or does not exist, then the point of the curve chart of a function with an abscissa x_0 is the inflection point.

Example 7.1. Study the graph of the function in order to find convexities and inflection points.

$$y = x^5 - x + 5.$$

Solution. Find the first and the second derivatives:

$y' = 5x^4 - 1$, $y'' = 20x^3$. The second derivative exists at all of the numerical axis $y'' = 0$ at $x = 0$.

$y'' > 0$ at $x > 0$ and $y'' < 0$ at $x < 0$. Therefore, the curve chart of the function $y = x^5 - x + 5$ on the interval $(-\infty;0)$ — is convex upward, while in the interval $(0;+\infty)$ — is convex downward. The point (0; 5) is the inflection point.

Finding areas of concavity and convexity of a function. Inflection points.

Example 7.2. Find inflection points and areas of concavity and convexity of the function:

$$f(x) = x \cdot e^{-x^2}.$$

Solution.

Let's find the first derivative of the function:

$$f'(x) = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = e^{-x^2} \cdot (1 - 2x^2).$$

Then find the second derivative of the function:

$$f''(x) = \left(e^{-x^2} \cdot (1 - 2x^2)\right)' =$$

$$= -2x \cdot e^{-x^2} \cdot (1 - 2x^2) + e^{-x^2} \cdot (-4x) =$$

$$= -2x \cdot (3 - 2x^2) \cdot e^{-x^2}.$$

Therefore,

$$f''(x) = 0$$

at $x_1 = 0$ and $x_{2,3} = \pm\sqrt{1.5}$.

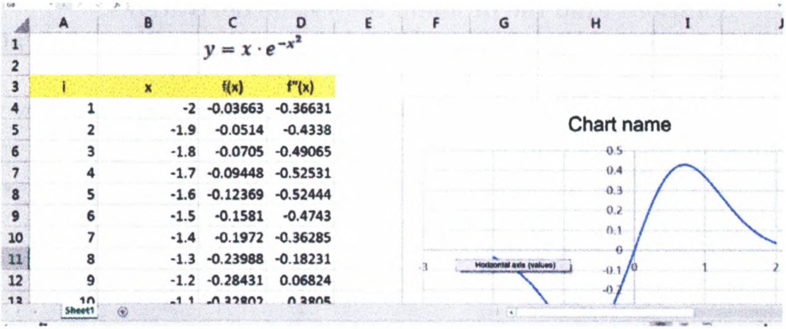
Let's find values of the function in an MS Excel worksheet:

$$f(x) = x \cdot e^{-x^2}$$

And its second derivative:

$$f''(x) = -2x \cdot (3 - 2x^2) \cdot e^{-x^2},$$

After that let's construct its graph (look at the Fig. 7.2, 7.3).



$y = x \cdot e^{-x^2}$

i	x	f(x)	f"(x)
1	-2	-0.03663	-0.36631
2	-1.9	-0.0514	-0.4338
3	-1.8	-0.0705	-0.49065
4	-1.7	-0.09448	-0.52531
5	-1.6	-0.12369	-0.52444
6	-1.5	-0.1581	-0.4743
7	-1.4	-0.1972	-0.36285
8	-1.3	-0.23988	-0.18231
9	-1.2	-0.28431	0.06824
10	-1.1	-0.32802	0.3805

Fig. 7.1. The initial worksheet.

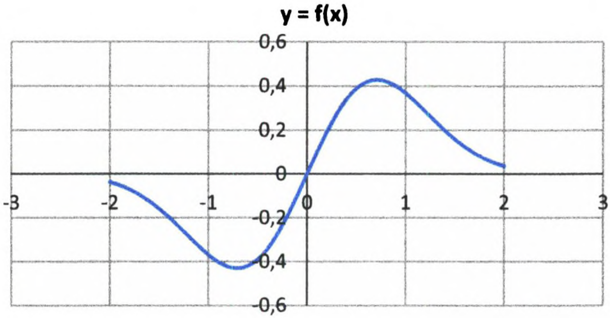


Fig. 7.2. The graph.

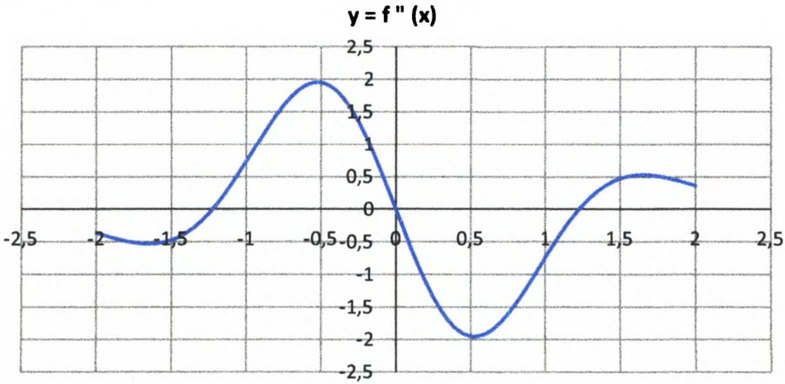


Fig. 7.3. The graph of the second derivative of the function.

It can be seen from the graphs that the marked points are inflection points. In the Fig. 7.4 there is a graph of the second derivative in the reduced scale.

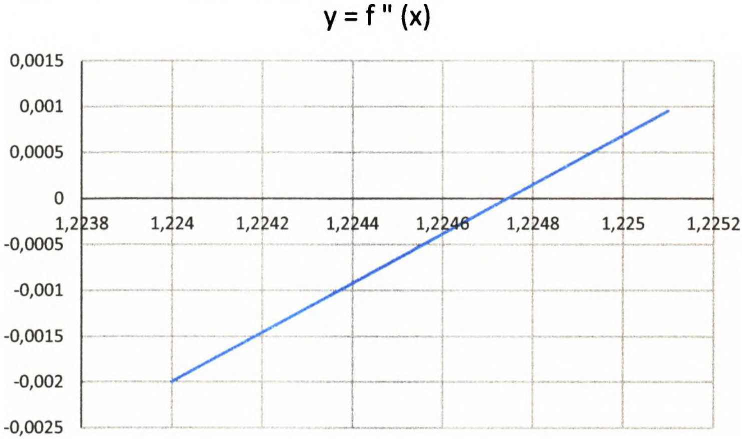


Fig. 7.4. The curve chart of the second derivative of the function.

Any of approximate formulas can be used to calculate the second derivative of a function while solving the assigned task.

	B	C	D	E	F	G	H	I	J
2								dx =	0.0000100000
3	x	x+dx	x+2dx	f(x+2dx)	f(x+dx)	f(x)	f''(x)		
4	1.224	1.22401	1.22402	0.273601	0.273605	0.27361	-0.001967315		
5	1.2241	1.22411	1.22412	0.273556	0.273561	0.273565	-0.001700862		
6	1.2242	1.22421	1.22422	0.273512	0.273516	0.273521	-0.001432188		
7	1.2243	1.22431	1.22432	0.273467	0.273472	0.273476	-0.001164624		
8	1.2244	1.22441	1.22442	0.273422	0.273427	0.273431	-0.000896505		
9	1.2245	1.22451	1.22452	0.273378	0.273382	0.273387	-0.000627831		0.00015
10	1.2246	1.22461	1.22462	0.273333	0.273338	0.273342	-0.000360822		
11	1.2247	1.22471	1.22472	0.273289	0.273293	0.273298	-9.32587E-05		0.0001
12	1.22471	1.22472	1.22473	0.273284	0.273289	0.273293	-6.4948E-05		
13	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		

Fig. 7.5. Calculating the second derivative of the function, using approximate formula.

Comparing the results of calculation applying the exact and approximate formulas, make sure that they match together (look at the Fig.7.6).

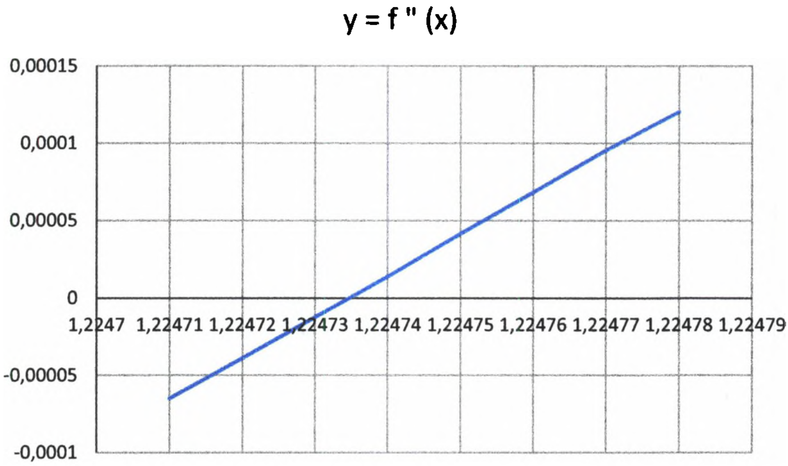


Fig. 7.6. The chart of the second derivative of the function.

Exercises for independent work

1. Find the inflection points and areas of concavity and convexity of the function:

$$y = e^{-x} \cdot \cos\cos x \ .$$

2. Find the inflection points and areas of concavity and convexity of the function:

$$y = \sqrt[3]{x+8} - \sqrt[3]{x-1}.$$

3. Find inflection points and areas of concavity and convexity of the function:

$$y = \frac{x^2 - x - 20}{x - 2}.$$

8. Absolute extrema

Introduction

Local (relative) extremum is a point in which function has the greatest (smallest) value in comparison with its values in all other points of the vicinity. An interval may contain many local extrema so you should be careful in order not to choose the first one as a solution of the task.

Global (absolute) extremum is a point in which function has bigger (smaller) value in comparison with any other point of the domain. Thus, global extremum is the optimal solution for the domain.

The task for a function of one variable can be set as follows. Let values of a variable x be concluded in an interval $[a; b]$. We call an *uncertainty interval* an interval of values x in which search of an optimum of criterion function is run. At the beginning of optimization process this interval has $b–a$ length. It is necessary to define values of an optimum of function with a margin error ε, that is to find in an interval $[a; b]$ point x, such that

$f(x - \varepsilon) < f(x) > f(x + \varepsilon)$ — when looking for maximum,

or

$f(x - \varepsilon) > f(x) < f(x + \varepsilon)$ — when looking for minimum.

Thus, we need to have the action plan which is inevitably leading to definition of a point x^* with an accuracy ε where this point wouldn't lie in the field of search. The methods considered further are also such action plans for search of optimum of functions.

Obviously, the most natural and easy way to narrow an interval of uncertainty for one-dimensional function is its division into several equal parts with the subsequent calculation of values of function in knots of the mesh.

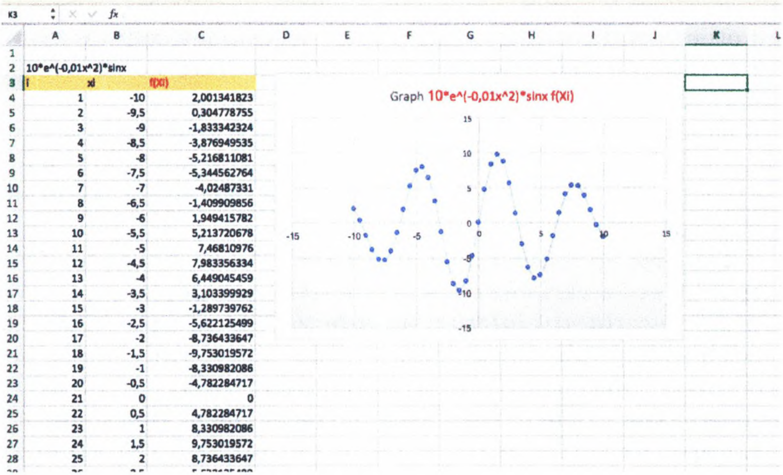
Example 8.1.

Find the absolute extrema of the function

$$f(x) = 10e^{-0{,}01x^2}\sin x.$$

Solution.

Let's plot a function graph using MS Excel. To do this, we look through the range B7:B107 where we set values of indew pendent variable, in the range C7:C107 we calculate correh sponding function values:



	A	B	C
1			
2	10*e^(-0,01x^2)*sinx		
3	i	xi	f(Xi)
4	1	-10	2,001341823
5	2	-9,5	0,304778755
6	3	-9	-1,833342324
7	4	-8,5	-3,876949535
8	5	-8	-5,216811081
9	6	-7,5	-5,344562764
10	7	-7	-4,02487331
11	8	-6,5	-1,409909856
12	9	-6	1,949415782
13	10	-5,5	5,213720678
14	11	-5	7,46810976
15	12	-4,5	7,983356334
16	13	-4	6,449045459
17	14	-3,5	3,103399929
18	15	-3	-1,289739762
19	16	-2,5	-5,622125499
20	17	-2	-8,736433647
21	18	-1,5	-9,753019572
22	19	-1	-8,330982086
23	20	-0,5	-4,782284717
24	21	0	0
25	22	0,5	4,782284717
26	23	1	8,330982086
27	24	1,5	9,753019572
28	25	2	8,736433647

Fig. 8.1.1. Finding the absolute extrema visually.

When analyzing the graph, we see that the global maximum is located in the neighborhood of point x=1,5. For more detailed definition of point of the global maximum in the range B7:B107 we should put between 1,5 and 1,6 such values as 1,51, 1,52, ..., 1,59. In the range C7:C107 compute cor; responding function values. Then we understand that derivative becomes 0 between values 1,543 and 1,544 (look at the fig. 8.1.2). Doing the same necessary number of times, you can get the extrema point value with needed accuracy. Process described above (step-by-step approximation) could be seen on figures from 8.1.2. till 8.1.9. As a result, we get the value of the global extremum: $x_0 = 1{,}54000594190$.

D31 fx =-0,2*B31*EXP(1)^(-0,01*B31^2)*SIN(B31)+10*EXP(1)^(-0,01*B31^2)*COS(B31)

	A	B	C	D
22	19	-1	-8,330982086	5,18264244
23	20	-0,5	-4,782284717	8,70609061
24	21	0	0	10
25	22	0,5	4,782284717	8,70609061
26	23	1	8,330982086	5,18264244
27	24	1,5	9,753019572	0,39904328
28	25	1,51	9,756510997	0,29924529
29	26	1,52	9,759004569	0,19947465
30	27	1,53	9,760500618	0,09974231
31	28	1,54	9,76099958	5,9214E-05
32	29	1,541	9,760994656	-0,009906
33	30	1,542	9,760979768	-0,0198705
34	31	1,543	9,760954916	-0,0298345
35	32	1,544	9,760920099	-0,0397978

Fig. 8.1.2. Specification of the value for the absolute extremum ($1{,}540 < x_0 < 1{,}541$).

D2 fx =-0,2*B2*EXP(1)^(-0,01*B2^2)*SIN(B2)+10*EXP(1)^(-0,01*B2^2)*COS(B2)

	A	B	C	D
1	i	xi	f(xi)	f'(xi)
2	1	1,50000000000	9,75301957246	0,39904
3	2	1,51000000000	9,75651099709	0,29925
4	3	1,52000000000	9,75900456946	0,19947
5	4	1,53000000000	9,76050061779	0,09974
6	5	1,54000000000	9,76099957982	0,00006
7	6	1,54010000000	9,76099953591	-0,00094
8	7	1,54020000000	9,76099939235	-0,00193
9	8	1,54030000000	9,76099914914	-0,00293
10	9	1,54040000000	9,76099880627	-0,00393
11	10	1,54050000000	9,76099836375	-0,00492
12	11	1,54060000000	9,76099782158	-0,00592
13	12	1,54070000000	9,76099717976	-0,00692
14	13	1,54080000000	9,76099643828	-0,00791
15	14	1,54090000000	9,76099559716	-0,00891

Fig. 8.1.3. Specification of the value for the absolute extremum ($1{,}5400 < x_0 < 1{,}5401$).

	A	B	C	D	E	F	G	H
1	I	1,53999500000	f(xi)	f'(xi)				
2	1	1,53999600000	9,76099957950	0,00010				
3	2	1,53999700000	9,76099957959	0,00009				
4	3	1,53999800000	9,76099957968	0,00008				
5	4	1,53999900000	9,76099957975	0,00007				
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific way of formatting			
7	6	1,54000100000	9,76099957987	4,92E-05				
8	7	1,54000200000	9,76099957992	3,92829E-05				
9	8	1,54000300000	9,76099957995	2,93175E-05				
10	9	1,54000400000	9,76099957997	1,9352E-05				
11	10	1,54000500000	9,76099957999	9,39E-06				
12	11	1,54000600000	9,76099957999	-5,7899E-07				
13	12	1,54000700000	9,76099957999	-1,05445E-05				

Fig. 8.1.4. Specification of the value for the absolute extremum ($1{,}54000 < x_0 < 1{,}54001$).

	A	B	C	D	E	F	G	H
1	i	xi	f(xi)	f'(xi)				
2	1	1,50000000000	9,75301957246	0,39904				
3	2	1,51000000000	9,75651099709	0,29925				
4	3	1,52000000000	9,75900456946	0,19947				
5	4	1,53000000000	9,76050061779	0,09974				
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific way of formatting			
7	6	1,54001000000	9,76099957991	-4,04E-05				
8	7	1,54002000000	9,76099957901	-0,0001401				
9	8	1,54003000000	9,76099957711	-0,0002398				
10	9	1,54004000000	9,76099957421	-0,0003394				
11	10	1,54005000000	9,76099957032	-0,0004391				
12	11	1,54006000000	9,76099956543	-0,0005387				
13	12	1,54007000000	9,76099955955	-0,0006384				

Fig. 8.1.5. Specification of the value for the absolute extremum ($1{,}540005 < x_0 < 1{,}540006$).

	A	B	C	D	E	F	G
3	2	1,53999700000	9,76099957959	0,00009			
4	3	1,53999800000	9,76099957968	0,00008			
5	4	1,53999900000	9,76099957975	0,00007			
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific way of		
7	6	1,54000100000	9,76099957987	4,92E-05			
8	7	1,54000200000	9,76099957992	3,92829E-05			
9	8	1,54000300000	9,76099957995	2,93175E-05			
10	9	1,54000400000	9,76099957997	1,9352E-05			
11	10	1,54000590000	9,76099957999	4,18E-07			
12	11	1,54000600000	9,76099957999	-5,79E-07			
13	12	1,54000610000	9,76099957999	-1,58E-06			

Fig. 8.1.6. Specification of the value for the absolute extremum ($1{,}5400059 < x_0 < 1{,}5400060$).

D11 f_x =-0,2*B11*EXP(1)^(-0,01*B11^2)*SIN(B11)+10*EXP(1)^(-0,01*B11^2)*COS(B11)

	A	B	C	D	E	F
3	2	1,53999700000	9,76099957959	0,00009		
4	3	1,53999800000	9,76099957968	0,00008		
5	4	1,53999900000	9,76099957975	0,00007		
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific w	
7	6	1,54000100000	9,76099957987	4,92E-05		
8	7	1,54000200000	9,76099957992	3,92829E-05		
9	8	1,54000300000	9,76099957995	2,93175E-05		
10	9	1,54000400000	9,76099957997	1,9352E-05		
11	10	1,54000594000	9,76099957999	1,89E-08		
12	11	1,54000595000	9,76099957999	-8,1E-08		
13	12	1,54000596000	9,76099957999	-1,8E-07		
14	13	1,54000597000	9,76099957999	-2,8E-07		
15	14	1,54000598000	9,76099957999	-3,8E-07		
16	15	1.54000599000	9.76099957999	-4.8E-07		

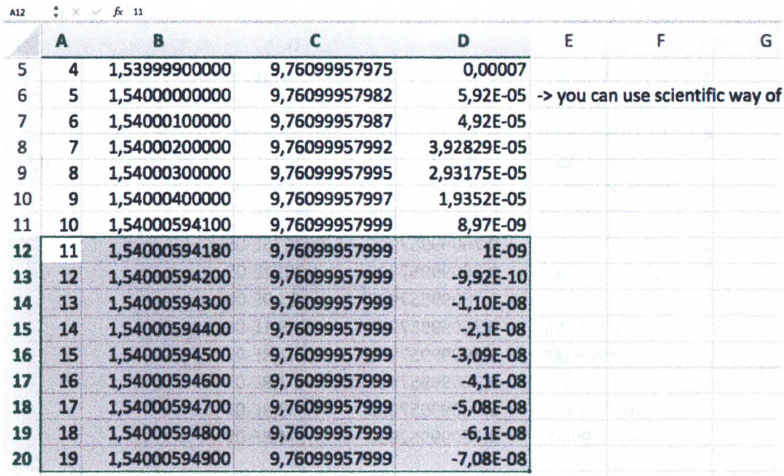
Fig. 8.1.7. Specification of the value for the absolute extremum ($1,54000594 < x_0 < 1,54000595$).

D17 f_x =-0,2*B17*EXP(1)^(-0,01*B17^2)*SIN(B17)+10*EXP(1)^(-0,01*B17^2)*COS(B17)

	A	B	C	D	E	F
3	2	1,53999700000	9,76099957959	0,00009		
4	3	1,53999800000	9,76099957968	0,00008		
5	4	1,53999900000	9,76099957975	0,00007		
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific way	
7	6	1,54000100000	9,76099957987	4,92E-05		
8	7	1,54000200000	9,76099957992	3,92829E-05		
9	8	1,54000300000	9,76099957995	2,93175E-05		
10	9	1,54000400000	9,76099957997	1,9352E-05		
11	10	1,54000594100	9,76099957999	8,97E-09		
12	11	1,54000594200	9,76099957999	-9,9E-10		
13	12	1,54000594300	9,76099957999	-1,10E-08		
14	13	1,54000594400	9,76099957999	-2,1E-08		
15	14	1,54000594500	9,76099957999	-3,09E-08		
16	15	1,54000594600	9,76099957999	-4,1E-08		
17	16	1,54000594700	9,76099957999	-5,08E-08		
18	17	1,54000594800	9,76099957999	-6,1E-08		
19	18	1,54000594900	9,76099957999	-7,08E-08		

Sheet1 Sheet2 +

Fig. 8.1.8. Specification of the value for the absolute extremum ($1,540005941 < x_0 < 1,540005942$).


A12 fx 11

	A	B	C	D	E	F	G
5	4	1,53999900000	9,76099957975	0,00007			
6	5	1,54000000000	9,76099957982	5,92E-05	-> you can use scientific way of		
7	6	1,54000100000	9,76099957987	4,92E-05			
8	7	1,54000200000	9,76099957992	3,92829E-05			
9	8	1,54000300000	9,76099957995	2,93175E-05			
10	9	1,54000400000	9,76099957997	1,9352E-05			
11	10	1,54000594100	9,76099957999	8,97E-09			
12	11	1,54000594180	9,76099957999	1E-09			
13	12	1,54000594200	9,76099957999	-9,92E-10			
14	13	1,54000594300	9,76099957999	-1,10E-08			
15	14	1,54000594400	9,76099957999	-2,1E-08			
16	15	1,54000594500	9,76099957999	-3,09E-08			
17	16	1,54000594600	9,76099957999	-4,1E-08			
18	17	1,54000594700	9,76099957999	-5,08E-08			
19	18	1,54000594800	9,76099957999	-6,1E-08			
20	19	1,54000594900	9,76099957999	-7,08E-08			

Fig. 8.1.9. Root value $x = 1{,}54000594190$.

Thus, when computing this certain example, it is possible to use either precise derivative values, which are calculated using this formula:

$$f'(x) = 10 \cdot (\cos x - 0{,}02 \cdot x \cdot \sin x) \cdot \exp(-0{,}01x^2),$$

or approximate, computed by the formula:

$$f'(x_k) \approx (f(x + \Delta x) - f(x)),$$

where $\Delta x = 0{,}000000001$.

One more way of solving this problem is based on using *method of tangents* (also called as Newton's method).

As

$$f'(x) = 10 \cdot (\cos x - 0{,}02 \cdot x \cdot \sin x) \cdot \exp(-0{,}01x^2),$$

equation $f'(x)=0$ is equal to

$$\cos x - 0{,}02 \cdot x \cdot \sin x = 0.$$

It is possible to take as the first approximation of the root of this equation $x = 1{,}5$ (the process of obtaining this value is depicted in fig. 8.1.1.)

Let $g(x) = \cos x - 0{,}02x \sin x$, so

$$g'(x) = -1.02 \sin x - 0.02 \cos x.$$

More detailed description of the algorithm of solving equaetions using method of tangents is written in previous chapters. Results could be seen in the table 8.1.

Table 8.1.

Finding the root f'(x)=0		By using method of tangents		
Xk	g (Xk)	g' (Xk)	Xk+1	dXk
1,50000000000	0,0408123520696	-1,019567002	1,540029103	0,040029103
1,54002910252	-0,0000236346008	-1,02046476	1,540005942	-2,31606E-05
1,54000594190	-0,0000000000003	-1,020464731	1,540005942	-3,193E-13
1,54000594190	0,0000000000001	1657,377773	1,540005942	0

From the table it is obvious that the value of the root with high accuracy is got by 3 iterations.

Exercises for independent work

1. Find absolute extrema of the function:

$$f(x) = x^4 + 5x^2 - 10x.$$

2. Find absolute extrema of the function:

$$f(x) = \sqrt[3]{x} \cdot e^{|x-2|}.$$

9. Numerical study of a function

Introduction

It is advisable to conduct the analysis of the function $y = f(x)$ in a certain order.

Algorithm

1. Find the domain of the function.
2. Find (if possible) the points of intersection of the graph with the coordinate axes.
3. Find the asymptotes of the function graph.
4. Find the extrema of the function and the intervals of monotonicity.
5. Find the intervals of convexity and inflection points of the function graph.

Based on the conducted study, draw the graph of the function. If the graph is not entirely clear, it is possible to plot several points of the graph additionally and identify other features of the function (periodicity etc.). Sometimes, it is expedient to accompany the operations of analysis with a gradual plot of the function graph.

Example 9.1. Investigate the function $y = \dfrac{x}{1 - x^2}$ and build its graph.

Solution.

1. The function is undefined at $x = 1$ and $x = (-1)$. Its domain consists of three intervals: $(-\infty; -1) \cup (-1; 1) \cup (1; +\infty)$, and the graph has three branches.

2. If $x = 0$, then $y = 0$, and the graph intersects the Ox and Oy axes at the point $(0; 0)$.

3. The lines $x = 1$ and $x = -1$ are vertical asymptotes. Try to find an oblique asymptote:

$$k = \lim_{x\to\infty} \frac{\frac{x}{1-x^2}}{x} = \lim_{x\to\infty} \frac{1}{1-x^2} = 0.$$

($k = 0$ at $x \to +\infty$ and at $x \to -\infty$),

$$b = \lim_{x\to\infty} \left(\frac{x}{1-x^2} - 0 \cdot x \right) = \lim_{x\to\infty} \frac{x}{1-x^2} = 0.$$

Therefore, there is a horizontal asymptote $y = 0$. The line $y = 0$ is an asymptote at $x \to +\infty$ and at $x \to -\infty$.

4. Find the intervals of increasing and decreasing of the function

$$y' = \left(\frac{x}{1-x^2} \right)' = \frac{1 \cdot (1-x^2) - x \cdot (-2x)}{(1-x^2)^2} = \frac{x^2+1}{(1-x^2)^2}.$$

In that case, $y' > 0$ in domain, and the function is increase ing on each interval of its domain.

The critical points are $x_1 = 1$ and $x_2 = -1$,but they do not belong to the domain of a function. The function has no extrema.

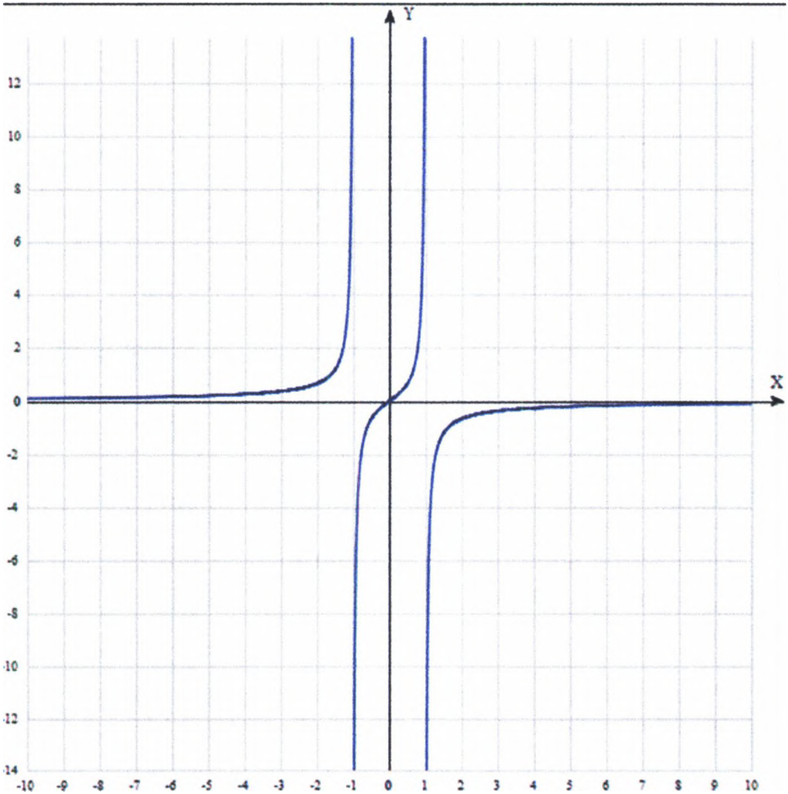
5. Examine the function for concavity (convexity). Find the second derivative:

$$y'' = \left(\frac{x^2+1}{(1-x^2)^2}, \right)' = \frac{2x(1-x^2)^2 - (x^2+1)2(1-x^2)(-2x)}{(1-x^2)^4} =$$

$$= \frac{2x(x^2+3)}{(1-x^2)^3}.$$

The second derivative equals zero or does not exist at the points $x_1 = 0$, and $x_2 = -1$, $x_3 = 1$. The point (0; 0) — is the inflection point of the graph. The graph is concave down in the intervals (–1; 0) and (1; + ∞); concave up in the intervals (–∞; –1) and (0; 1).

Let's draw the function graph:



Example 9.2.

Perform a full study of the function and graph it:

$$f(x) = \frac{x^2 - 2x + 2}{x - 1}.$$

Solution.

1. The domain of the function $D(f) = (-\infty; 1) \cup (1; +\infty)$, since the formula that defines the function is meaningful for all values of x except for the point $x = 1$, which is an infinity discontinuity point of the function. Identically,

$$\lim_{x \to 1-0} \frac{x^2 - 2x + 2}{x - 1} = -\infty;$$

since the numerator takes a positive value in the vicinity of $x = 1$ and the denominator tends x to zero, in the left side, remaining negative. Similarly,

$$\lim_{x \to 1+0} \frac{x^2 - 2x + 2}{x - 1} = +\infty.$$

In that case, $x = 1$ is the vertical asymptote.

2. The only point of intersection of the function graph with the axis Oy: (0; -2). Obviously, there are no intersections with the axis Ox, since the numerator of the fraction is strictly positive.

3. We will investigate the function for the presence of oblique asymptotes using MS Excel. In the range B4: B20, we will place values of the independent variable from 1000000000 to 10000000000, in the range C4: C20 the corresponding values of the function, in the range D4: D20, we will calculate values of the coefficient $k = \dfrac{f(x)}{x}$, in the range E4: E20, we will calculate the values of the coefficient $b = f(x) - kx$. As can be seen from the calculations, $a = 1$, $b = -1$. In that case, the equation of the right oblique asymptote: $y = x - 1$.

Similarly, we can derive the equation of the left oblique asymptote:

$$y = x - 1.$$

	A	B	C	D	E	F	G
1							
2							
3	1	x	f(x)	a	b		
4		1E+09	1E+09	1	-1		
5		1E+09	1E+09	1	-1		
6		1E+09	1E+09	1	-1		
7		1E+09	1E+09	1	-1		
8		1E+09	1E+09	1	-1		
9		1E+09	1E+09	1	-1		
10		1E+09	1E+09	1	-1		
11		1E+09	1E+09	1	-1		
12		1E+09	1E+09	1	-1		
13		1E+09	1E+09	1	-1		
14		1E+09	1E+09	1	-1		

Fig. 9.1. Finding the equation of the oblique asymptote.

4. Calculate the first derivative of the function:

$$y' = \frac{x(x-2)}{(x-1)^2}.$$

From the last formula it follows that the 1st derivative equals zero at the points: $x_1 = 0$; $x_2 = 2$. Using MS Excel, we determine whether the function has extrema at these points.

x	f(x)	f'(x)
1.5	2.5	-5
1.55	2.36818	-3.52292
1.6	2.26667	-2.51852
1.65	2.18846	-1.81293
1.7	2.12857	-1.30321
1.75	2.08333	-0.92593
1.8	2.05	-0.64062
1.85	2.02647	-0.42072
1.9	2.01111	-0.24829
1.95	2.00263	-0.11095
2	2	0.88E-16
2.05	2.00238	-0.09081
2.1	2.00909	0.166041
2.15	2.01957	0.229062

Fig. 9.2. Building a graph in a vicinity of critical points.

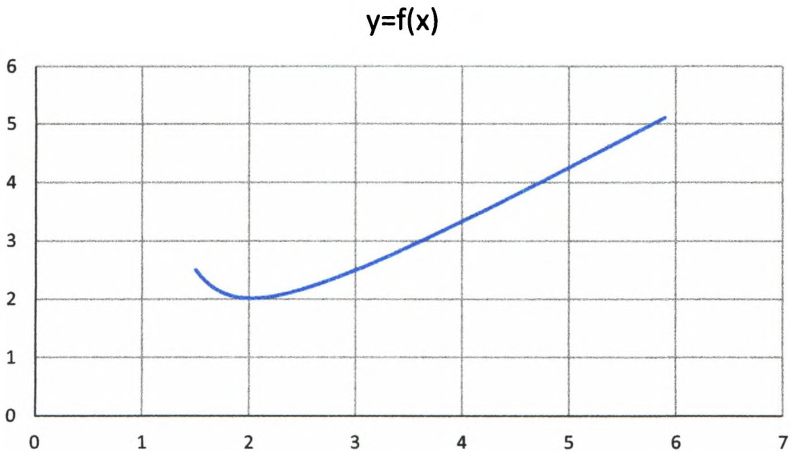


Fig. 9.3. Graph of the function near the extremum point.

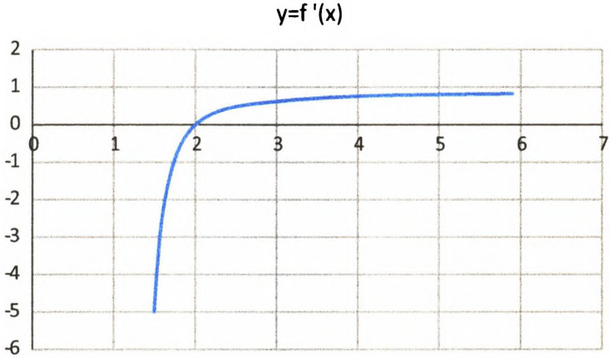


Fig. 9.4. Graph of the derivative of the function in a vicinity of the extremum point.

As can be seen from the graphs given at the point $x_2 = 2$, the function reaches its minimum at the point of $f(2) = 2$. Similarly, it can be shown that at the point $x_1 = 0$ the function reaches a minimum at the point of $f(0) = -2$.

5. Calculate the second derivative of the function:

$$y'' = \frac{2(x-3)}{(x-1)^2}.$$

The second derivative appeals to zero at the point $x = 3$. Using MS Excel, we will determine whether this point is a point of inflection for this function.

	A	B	C	D
4	i	x	f(x)	f`(x)
5		1.5	2.5	-12
6		1.6	2.26667	-7.77778
7		1.7	2.12857	-5.20612
8		1.8	2.05	-3.75
9		1.9	2.01111	-2.71605
10		2	2	-2
11		2.1	2.00909	-1.4876
12		2.2	2.03333	-1.11111
13		2.3	2.06923	-0.8284
14		2.4	2.11429	-0.61224
15		2.5	2.16667	-0.44444
16		2.6	2.225	-0.3125

Fig. 9.5. Calculation of the inflection point.

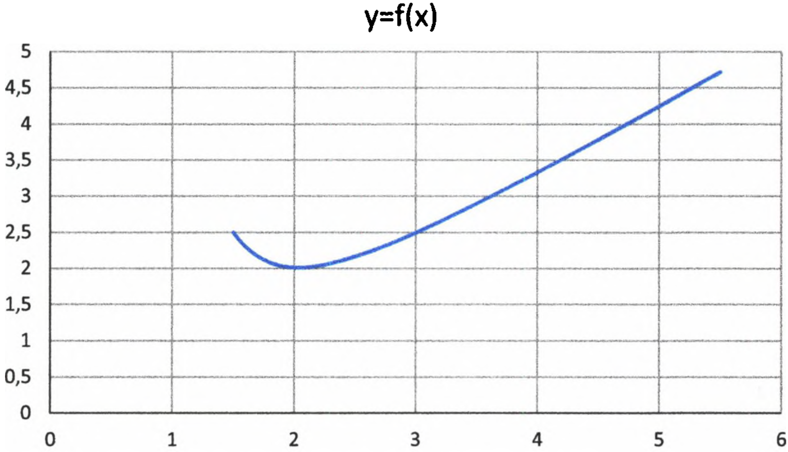


Fig. 9.6. Building the function graph in the vicinity of inflection point.

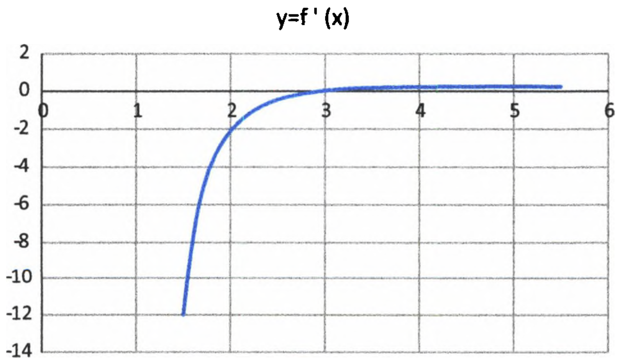


Fig. 9.7. Plotting the derivative of the function in the vicinity of inflection point.

As can be seen from the provided graphs, the point $x = 3$ is the inflection point of the function. Obviously, that the graph of the function is "divided" into two parts relatively to the vertical asymptote $x = 1$. Plot the graphs of the "left" and "right" parts of the function.

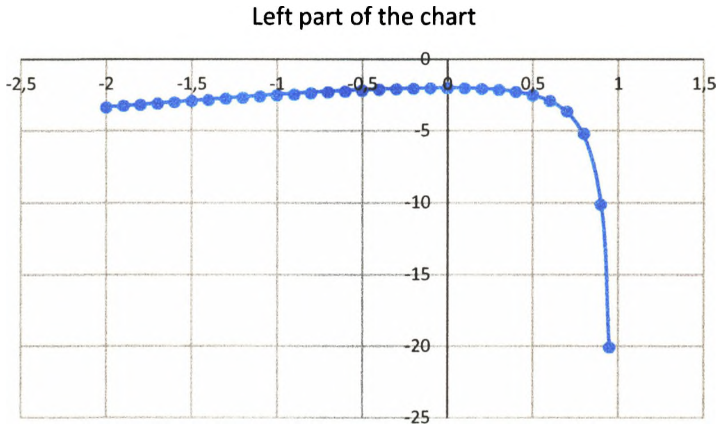


Fig. 9.8. Line with markers of the left side relatively to the vertical asymptote

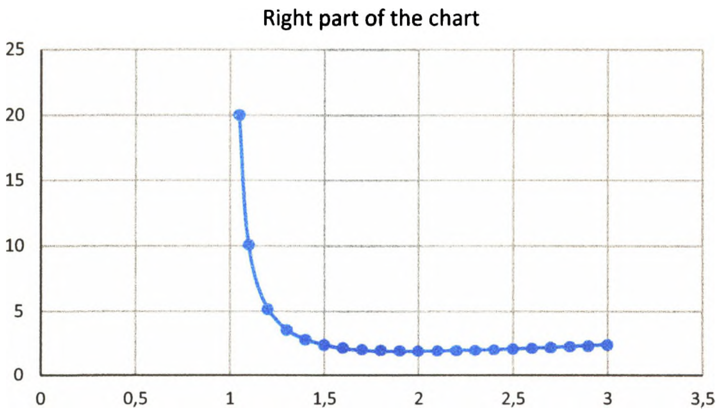


Fig. 9.9. Line with markers of the right side relatively to the vertical asymptote.

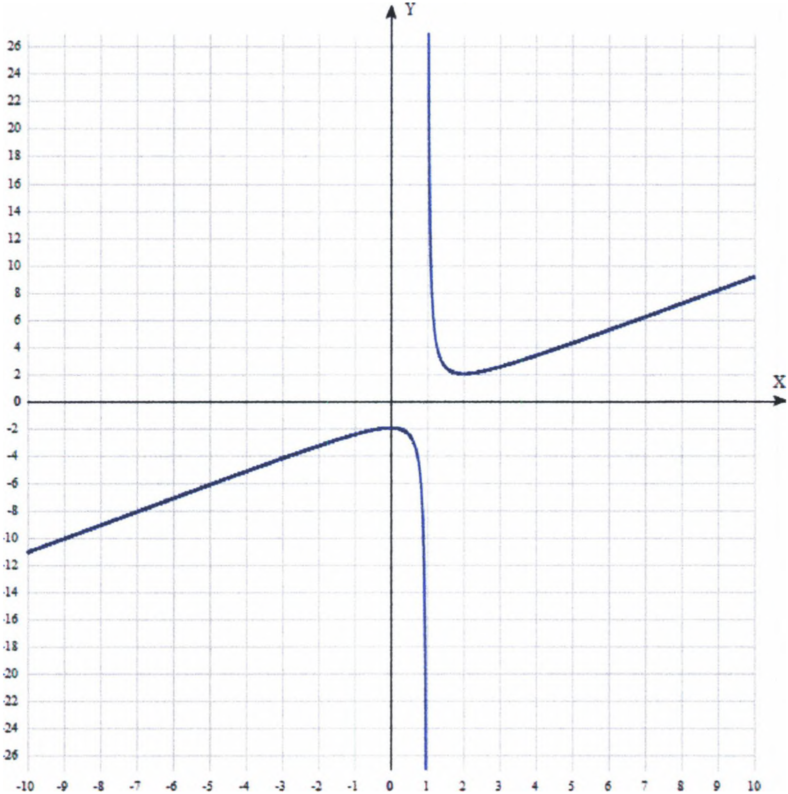


Fig. 9.10. General view of the function graph.

Exercises for independent work

1. Conduct a complete study of the function and plot its graph: $y = e^{-x} \cos x$.

2. Conduct a complete study of the function and plot its graph: $y = \sqrt[3]{x+8} - \sqrt[3]{x-1}$.

3. Conduct a complete study of the function and plot its graph:

$$y = \frac{x^2 - x - 20}{x - 2}.$$

10. Approximation by Taylor expansion

Introduction

It is known from calculus, that if a function $f(x)$ has $(n+1)$ derivative in some point's neighborhood, than here the Taylor's formula with center of the expansion $x = a$ is held:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$
$$\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x), \qquad (10.1)$$

where $R_{n+1}(x)$ — residual term, practically always approaches zero with increasing n.

In the particular case when $a = 0$ formula (10.1) is called the Maclaurin's formula:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$
$$\ldots + \frac{f^{(n)}(0)}{n!}x^n + R_{n+1}(x). \qquad (10.2)$$

Right part of decomposition (10.2) is polynomial of n-th degree

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n,$$

which, if we forget about residual term, approximates initial function $f(x)$ in zero neighborhood:

$$f(x) \approx P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n \qquad (10.3)$$

Remark. If we leave in decomposition (10.1) only the components up to the first derivative, then in fact we get function approximation by its tangent, which we have already considered it the previous chapter:

$$f(x) \approx f(a) + f'(a)(x - a).$$

Example 10.1. Consider $f(x) = \sin x$. Its derivatives in 0 are following:

$f^{(1)}(x) = \cos(x) \Rightarrow f^{(1)}(0) = 1;$

$f^{(2)}(x) = -\sin(x) \Rightarrow f^{(2)}(0) = 0;$

$f^{(3)}(x) = -\cos(x) \Rightarrow f^{(3)}(0) = -1;$

$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = 0.$

Then derivatives always repeat as in a circle. Then follows, that when $x = 0$, derivatives of even order are equal to 0, then odd derivatives change their sign from "+" to "–".

Using formula (10.3), we get an approximate equality:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}.$$

Approximate $\sin x$ in neighborhood of zero by polynomial of the first degree $P_1(x) = x$, so we get

$$\sin x \approx x.$$

Let's create MS Excel workbook «Maclaurin», and on a separate worksheet let's fill out three columns. First of them is an argument value (let's say, with step 0,1 in the interval [-2,2]), second column — with according values of sine function, which could be calculated by MS Excel function "SIN", and the third column — corresponding values of identical linear function $f(x) = x$. Let's enter formula in higher row cell, stretch the values of the column down, getting values in the cells quickly (the technique «drag and drop» is one of the most essential features in MS Excel). We highlight all three columns using *INSERT – RECOMMENDED CHARTS – LINE,* getting two lines on the same axises: blue colour for *y=sin(x)* and orange colour for $P_1(x) = x$. Evidently, graphs differ too much, approximation couldn't be called «good enough» (Fig 10.1).

x	sin x	P1(x)
-2,0	-0,909	-2,0
-1,9	-0,946	-1,9
-1,8	-0,974	-1,8
-1,7	-0,992	-1,7
-1,6	-1,000	-1,6
-1,5	-0,997	-1,5
-1,4	-0,985	-1,4
-1,3	-0,964	-1,3
-1,2	-0,932	-1,2
-1,1	-0,891	-1,1
-1,0	-0,841	-1,0
-0,9	-0,783	-0,9
-0,8	-0,717	-0,8
-0,7	-0,644	-0,7
-0,6	-0,565	-0,6
-0,5	-0,479	-0,5
-0,4	-0,389	-0,4
-0,3	-0,296	-0,3
-0,2	-0,199	-0,2
-0,1	-0,100	-0,1
0,0	0,000	0,0
0,1	0,100	0,1
0,2	0,199	0,2
0,3	0,296	0,3
0,4	0,389	0,4
0,5	0,479	0,5

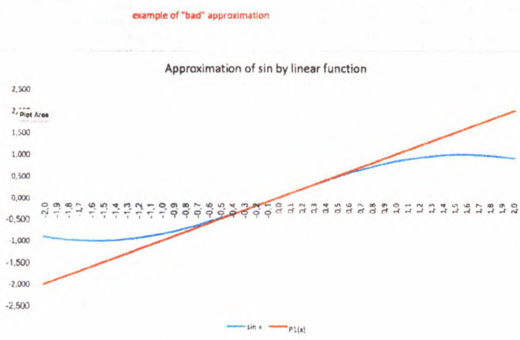


Fig. 10.1.

Now we are going to approximate $\sin x$ in a neighborhood of 0 by cubic polynomial $P_3(x) = x - \dfrac{x^3}{3!}$:

$$\sin x \approx x - \frac{x^3}{3!}.$$

Let's create a new worksheet and repeat all steps as before:

x	sin x	P3(x)
-2,0	-0,909	-0,667
-1,9	-0,946	-0,757
-1,8	-0,974	-0,828
-1,7	-0,992	-0,881
-1,6	-1,000	-0,917
-1,5	-0,997	-0,938
-1,4	-0,985	-0,943
-1,3	-0,964	-0,934
-1,2	-0,932	-0,912
-1,1	-0,891	-0,878
-1,0	-0,841	-0,833
-0,9	-0,783	-0,779
-0,8	-0,717	-0,715
-0,7	-0,644	-0,643
-0,6	-0,565	-0,564
-0,5	-0,479	-0,479
-0,4	-0,389	-0,389
-0,3	-0,296	-0,296
-0,2	-0,199	-0,199
-0,1	-0,100	-0,100
0,0	0,000	0,000
0,1	0,100	0,100

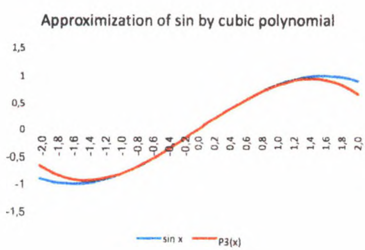
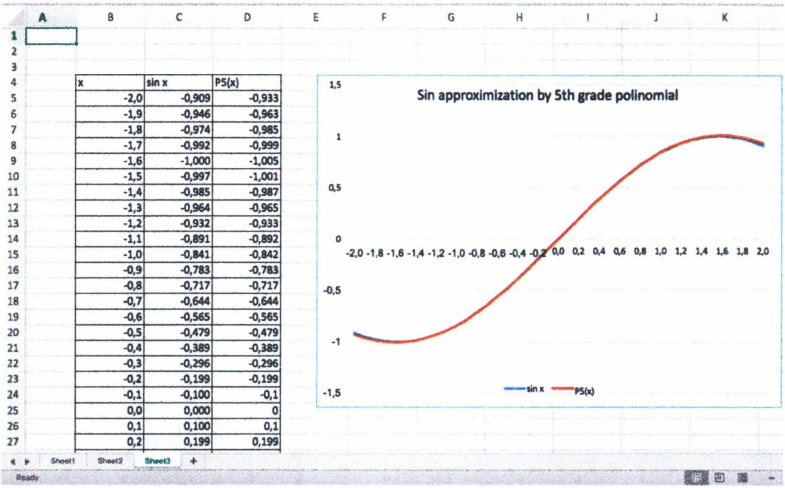


Fig. 10.2. Approximation of $\sin(x)$ by a cubic polynomial

Obviously, approximation is much better now, functions differ only at their tails. Well, let's approximate $\sin(x)$ in some neighborhood of 0 by 5$^{\text{th}}$ degree polynomial

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

We are creating a new worksheet and repeat the same steps as before. We get:



x	sin x	P5(x)
-2,0	-0,909	-0,933
-1,9	-0,946	-0,963
-1,8	-0,974	-0,985
-1,7	-0,992	-0,999
-1,6	-1,000	-1,005
-1,5	-0,997	-1,001
-1,4	-0,985	-0,987
-1,3	-0,964	-0,965
-1,2	-0,932	-0,933
-1,1	-0,891	-0,892
-1,0	-0,841	-0,842
-0,9	-0,783	-0,783
-0,8	-0,717	-0,717
-0,7	-0,644	-0,644
-0,6	-0,565	-0,565
-0,5	-0,479	-0,479
-0,4	-0,389	-0,389
-0,3	-0,296	-0,296
-0,2	-0,199	-0,199
-0,1	-0,100	-0,1
0,0	0,000	0
0,1	0,100	0,1
0,2	0,199	0,199

Fig. 10.3. Approximation of $\sin(x)$ by 5$^{\text{th}}$ degree polynomial

Well, now we see that the approximation is pretty good, graphs are about the same (have a slight difference at the ends of the interval).

Exercises for independent work

1. Expand $f(x) = e^{-x}$ according to the Maclaurin's formula. Using MS Excel, approximate this function by partial sums of Maclaurin's series — using polynomials of 1,2,3,4,5 degree. Consider intervals $[-1; 1]$ and $[-3; 3]$ with steps 0,1 and 0,03. For which intervals and steps approximation error is the most obvious?

2. Expand $f(x) = \ln(1 + x)$ according to the Maclaurin's formula. Using MS Excel, approximate this function by partial sums of Maclaurin's series — using polynomials of 1,2,3,4,5 degree. Consider intervals [–0,5; 0,5] with steps 0,05 and 0,01. *What is the area of convergency?

3. Expand $f(x) = \cos(x)$ according to the Maclaurin's formula. Using MS Excel, you can approximate this formula by particular sums of Maclaurin's terms — using polynomials of 2,4,6 degree. Consider the interval [–4; 4] with steps 0,04 and 0,01. *What is the area of convergency?

4*. Calculate the value of number e, expanding the function e^x using Maclaurin's formula till 20th degree and then substitute the point $x = 1$. Compare the result with function «EXP» or an ordinary calculator.

5. Graph of the sin(x) function on several periods, for instance, on the interval $[0;6\pi]$. What degree of Taylor's polynomial $P_n(x)$ should be taken, to have *sin(x)* be well approximated overall chosen interval $[0;6\pi]$?

11. Linear Programming

Introduction

In general, the linear programming problem (LP problem) can be written as:

$$Z(x) = c_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n \to \max(\min) \quad (11.1)$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1, \\ \ldots \\ a_{l_1}x_1 + a_{l_2}x_2 + \ldots + a_{l_n}x_n \le b_l, \\ a_{(l+1)_1}x_1 + a_{(l+1)_2}x_2 + \ldots + a_{(l+1)_n}x_n \le b_{(l+1)}, \\ \ldots \\ a_{m_1}x_1 + a_{m_2}x_2 + \ldots + a_{m_n}x_n \le b_m \end{cases} \quad (11.2)$$

$$x_j \ge 0;\ j = 1,2,\ldots t;\ t \le n \quad (11.3)$$

This entry means: find the extremum of the objective function (11.1) and its corresponding variables $X = (x_1,\ldots,x_m)$ satisfying the system of constraints (11.2) and non-negative conditions (11.3). Non-negative conditions are also called *trivial constraints*.

To solve the problem of linear programming by means of MS Excel, it is necessary to do the following:

1. Create a screen form to enter the condition of the task: variables, *objective function* (OF), constraints, boundary conditions;
2. Introduce the raw data into the screen form: OF coefficients, variable coefficients in constraints, right-hand parts of restrictions;
3. To introduce the dependencies from the mathematical model into the screen form: a formula for calculating the OF, formulas for calculating the values of the left-hand parts of

the constraints; set the OF: a cell with the objective function and the direction of optimization of the OF;

4. Enter limitations and boundary conditions: cells with values of variables, boundary conditions for tolerable values of variables, relations between right and left parts of constraints;
5. Set the parameters for solving the problem;
6. Run the Solver to get the solution;
7. Select the output format of the solution.

Example 11.1. Find the solution of the following linear programming problem.

$$f(x_1, x_2, x_3, x_4) = x_1 + 3x_2 + 2x_3 + 4x_4 \to \max$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 + x_4 \le 150, \\ 4x_1 + 5x_2 + 2x_3 + 2x_4 \le 100, \\ 5x_1 + x_2 + 3x_3 + 2x_4 \le 300, \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0. \end{cases}$$

It is necessary to create a screen form for typing the problem statement.

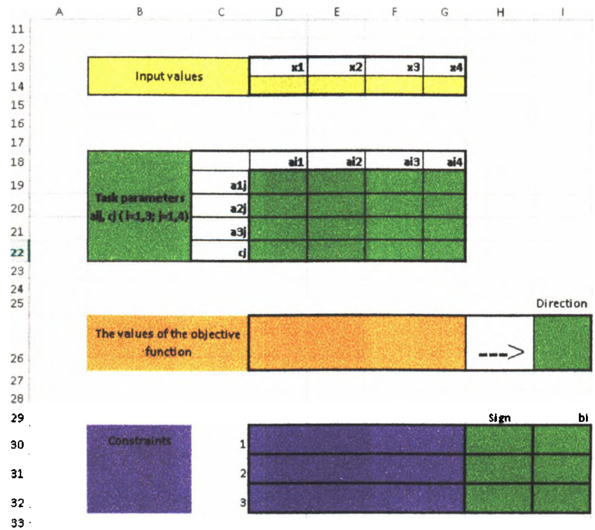


Fig. 11.1.

In the presented form each variable and each coefficient of the task is set in accordance with the MS Excel cell:

- ✓ The variables x_1, x_2, x_3, x_4 correspond to the cells D14, E14, F14, G14;
- ✓ The coefficients of the objective function $f(x_1, x_2, x_3, x_4)$ correspond to the cells D22, E22, F22, G22;
- ✓ The coefficients of the constraints correspond to the cells D19, E19, F19, G19, D20, E20, F20, G20, D21, E21, F21, G21;
- ✓ Right parts of the constraints correspond to the cells I30, I31, I32.

Then the input of the initial data into the screen form is carried out: OF coefficients, coefficients at variables in constraints, right parts of constraints.

Fig. 11.2.

In the cell D26, which will display the OF value, it is necessary to input the formula

$$f(x_1, x_2, x_3, x_4) = 5x_1 + 3x_2 + 2x_3 + 4x_4 \rightarrow \max,$$

by which this value will be calculated.

Using the designation of the corresponding cells in Excel, the formula for calculating the OF can be written as the sum of the products of each of the cells, assigned to the variables of the task (D14, E14, F14, G14), on the corresponding cells, allocated for the OF coefficients (D22, E22, F22, G22, G22) as follows:

- ✓ Place the cursor in cell D26;
- ✓ Click the button «f_x» and call out the window «Insert function»;
- ✓ Select the category «Math&Trig»;
- ✓ In the table «Choose a function» select the function «SUMPRODUCT» and click the button «OK»;
- ✓ In the appeared window «SUMPRODUCT», in the line «Array 1» you need to input expression D$14:G$14, and in the line «Array 2» — expression D22:G22.

After inputting the cells into the lines «Array 1» and «Array 2» in the window «SUMPRODUCT», there will appear numerical values of typed-into arrays (see fig. 11.3) and in the screen form in cell D26 will display the current value, calculated by the entered formula, that is 0.

The symbol "$" before the line number means that when you copy this formula to other places in the Excel sheet, the line number 14 will not change.

The symbol ":" means that all cells between the cells indicated on the left and right of the colon will be used in the formula (for example, B10:E10 indicates cells B10, C10, D10 and E10). After that, the target cell will have 0 (zero).

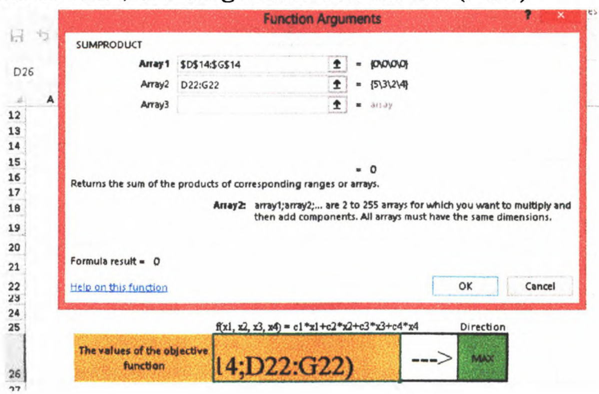


Fig. 11.3.

The left-hand parts of the problem constraint are the sum of the products of each cell, allocated for the values of the variables of the task (D14, E14, F14, G14) to the corresponding cells allocated for the coefficients of the specific constraint:

- ✓ D19, E19, F19, G19 — constraint 1;
- ✓ D20, E20, F20, G20 — constraint 2;
- ✓ D21, E22, F23, G24 — constraint 3.

The formulas corresponding to the left-hand side of the restriction are presented in the table.

Table 1

Left parts of constraints	Formula Excel
$3x_1 + 2x_2 + x_3 + x_4$	=SUMPRODUCT(D19:G19;$D$14:$G$14)
$4x_1 + 5x_2 + 2x_3 + 2x_4$	=SUMPRODUCT(D20:G20;$D$14:$G$14)
$5x_1 + x_2 + 3x_3 + 2x_4$	=SUMPRODUCT(D21:G21;$D$14:$G$14)

Formulas, specifying the left-hand parts of the problem's constraints differ from each other and from the formula in target cell D26 only by the line number in the first array. The row number of the second array is defined by the row, in which the restriction is written in the screen form. That is why, it is sufficient to copy the formula from the target cell to the cells on the left of the constraint to specify the dependencies for the left.

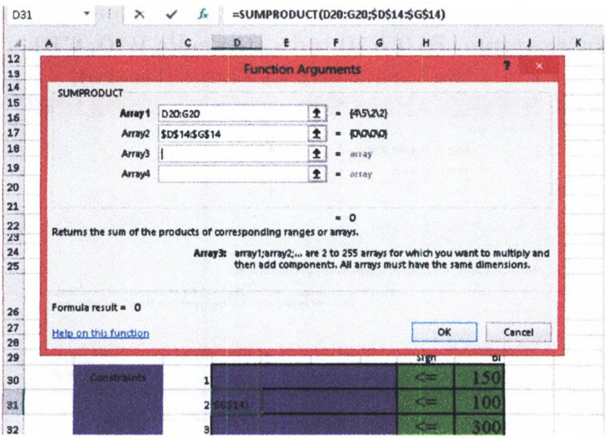


Fig. 11.4.

For further stages of solving the problem, it is necessary to install the MS Excel "Solver" add-in, as follows.

1. In the "Option" menu, select "Add-ins";
2. In the "Available add-ins" box, select the "Solver" check box, and then click "OK". If the "Solver" add-in is missing from the "Available Add-ins" list, click "Browse" to find it. If you receive a message that the "Solver" add-ins is not installed on your computer, click "Yes" to install it;
3. After downloading the "Solver" add-in, the "Solver" button becomes available on the "Data" tab.

After the configuration is complete, you must call " Solver" and do the following:

1. Put the cursor in the field "Optimize the objective function";
2. Enter the target cell address $D$26 or make one left click on the target cell in the on-screen form — this will be equivalent to entering the address from the keyboard;
3. Enter the optimization direction of the OF, by left-clicking on the "Max" selector button once.

In the "Solver" box in the "by changing Variable Cells" field, enter $D$14:$G$14 addresses. The required addresses can be entered into the "Change cells" field and automatically by selecting the corresponding cells of the variables directly in the screen form.

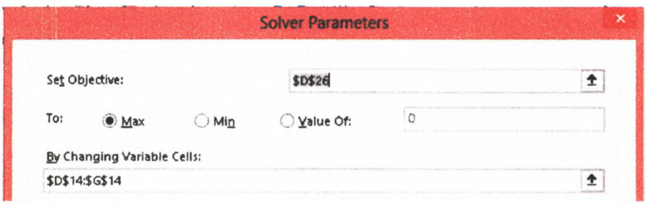


Fig. 11.5.

In the field «Subject to the constraints» you must select the button «Add», which is followed by appearing the window «Add Constraint».

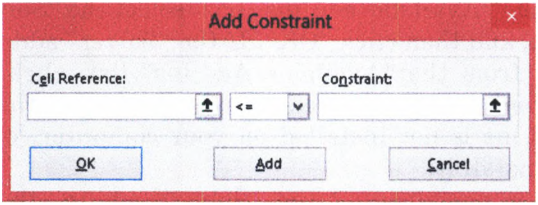


Fig. 11.6.

In the field «Cell Reference», enter the cell address of the constraint $D$30. It can be done both with the keyboard, and by the mouse highlighting all cells of variables directly in the screen form. In the sign box, open the list of suggested characters and select “<=”. In the “Constraint” box, enter a cell address of $I$30.

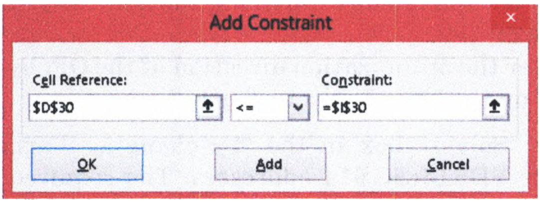


Fig. 11.7.

Similarly, you should enter the $D$31<=$I$31 and $D$32<=$I$32 constraints, and then confirm all the above conditions by pressing the “OK” button.

The “Solver” window after you enter all the parameters of the task is presented below. If you want to modify or remove some constraints or boundary conditions when you enter a task condition, you do so use the “Edit” or “Delete” buttons.

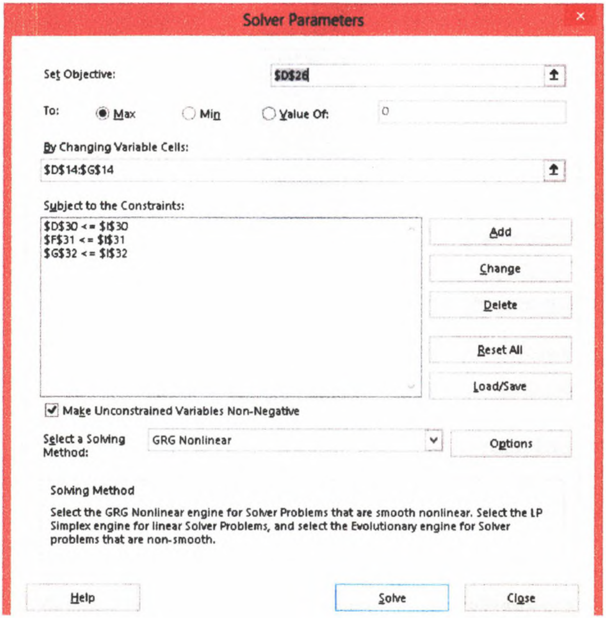


Fig. 11.8.

In the condition of the task there are constraints $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$, it is necessary to set the flag in «Make Unconstrained Variables Non-negative» box in the «Solver Parameters». In the window "Select a Solving Method" you shoud choose the option "Simplex LP".

After you enter the whole task condition, you should start the solution process by pressing the button "Solve". In the appeared window «Solver Results» you should select "Keep solver solution" and press "Ok"

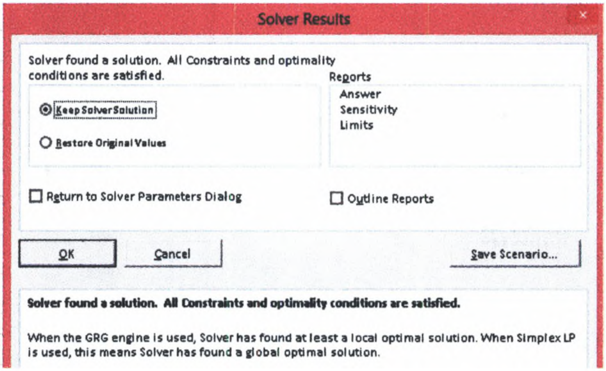


Fig. 11.9.

After all the described actions in the screen form will be the optimal solution of the task and the optimal value of the objective function.

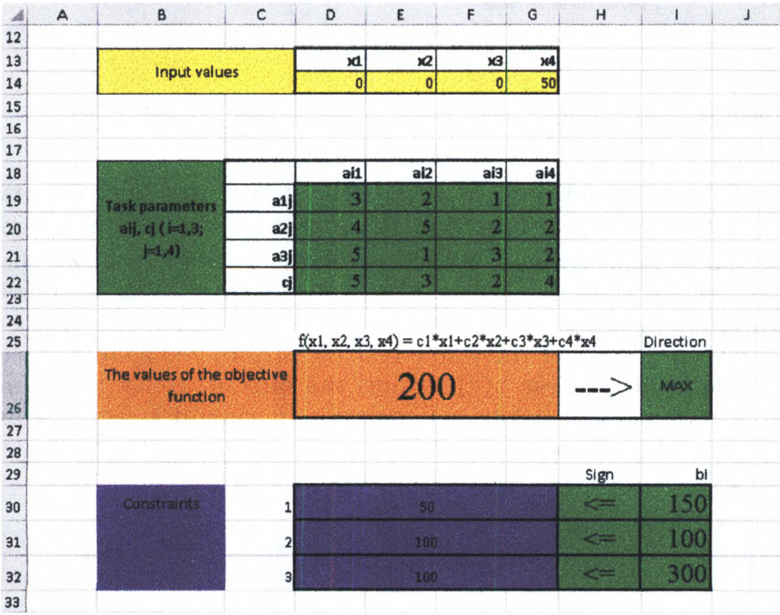


Fig. 11.10.

Remark. If the condition of the task assumes a number of variables, press the "Add" button in the "Solver" window and specify in the window that appears:

- In the "Cell reference" field, the address of the cells of the task variables, with the integer requirement.
- In the field of the constraint sign input «int»;
- Push the button «OK»;

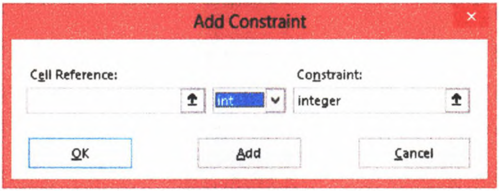


Fig. 11.11.

Example 11.2. The enterprise needs to select 5 business partners to sign a contract for the supply of goods worth up to 5 million rubles, determining the volume of the transaction with each of the partners and ensuring the maximum profit, taking into account that the expected amount of risks from the transactions will not exceed the amount of expected profit. The profit value t_i, risk r_i, $i = \overline{1,5}$ and the maximum amount of the m_i transaction with each of the partners is shown in the table.

Table 2

Option	Business-partners				
	ООО «Vector»	ОАО «Lutch»	ИП «Semenov»	ООО «Fenix»	ООО «Vostok»
m_i, rub	1 000 000	1 450 000	500 000	1 320 000	1 780 000
t_i, %	15,5	12,2	10,6	16	11,1
r_i,%	13,2	7,34	10,34	8,4	10,3

Let's create a mathematical model of the problem. Let x_i — the sum of the transaction with the i-m partner, S — the total possible amount of the transaction, then the objective function is the sum of the products of profit for each business partner profit, will be written as follows:

$$F = \frac{1}{100}\sum_{i=1}^{5} t_i x_i \to max .$$

Constraints (non-trivial and trrivial):

$$\begin{cases} \sum_{i=1}^{5} x_i \le S, \\ \sum_{i=1}^{5} r_i x_i \le \sum_{i=1}^{5} t_i x_i, \\ 0 \le x_i \le m_i, \; i = \overline{1,5}. \end{cases}$$

You must create a screen form to enter the task condition:

- ✓ Cells B31:F33 enter data from the table in the task condition;
- ✓ Cell values B37:F37 correspond to variables x_i, $i = \overline{1,5}$;
- ✓ Cell values B38:F38 reflect the lower constraint from the problem model
$$0 \le x_i \le m_i, \; i = \overline{1,5};$$
- ✓ Cell B42 contains the objective task function;
- ✓ Cell B44 contains the total amount of the transaction possible;
- ✓ Cells C48:E48 meet the constraints $\sum_{i=1}^{5} x_i \le S$;
- ✓ Cells C49:E49 correspond to the constraints $\sum_{i=1}^{5} r_i x_i \le \sum_{i=1}^{5} t_i x_i$.

	A	B	C	D	E	F
29		Business-partners				
30	Parameter	ООО «Vector»	OAO «Lutch»	ИП «Semenov»	ООО «Fenix»	ООО «Vostok»
31	m_i, rub	1 000 000	1 450 000	500 000	1 320 000	1 780 000
32	t_i, %	15,5	12,2	10,6	16	11,1
33	r_i,%	13,2	7,34	10,34	8,4	10,3
34						
35						
36	Changeable cells	x1	x2	x3	x4	x5
37		0	0	0	0	0
38	Lower value	0	0	0	0	0
39	Upper value	1 000 000	1450000	500000	1320000	1780000
40						
41						
42	The OF value	0	--->	max		
43						
44	S=	5000000				
45						
46						
47			Left part	Sign	Right part	
48	Constraints	1	0	<=	5000000	
49		2	0	<=	0	

Fig. 11.12.

The formula corresponding to the left side of the constraint and the objective function are shown in the table.

Table 3

Cell	Symbolic model formula	Formula Excel
B42	$\sum_{i=1}^{5} t_i x_i$	SUMPRODUCT(B37:F37;B32:F32)/100
C48	$\sum_{i=1}^{5} x_i$	SUM(B37:F37)
C49	$\sum_{i=1}^{5} r_i x_i$	SUMPRODUCT(B33:F33;B37:F37)
E49	$\sum_{i=1}^{5} t_i x_i$	SUMPRODUCT(B32:F32;B37:F37)

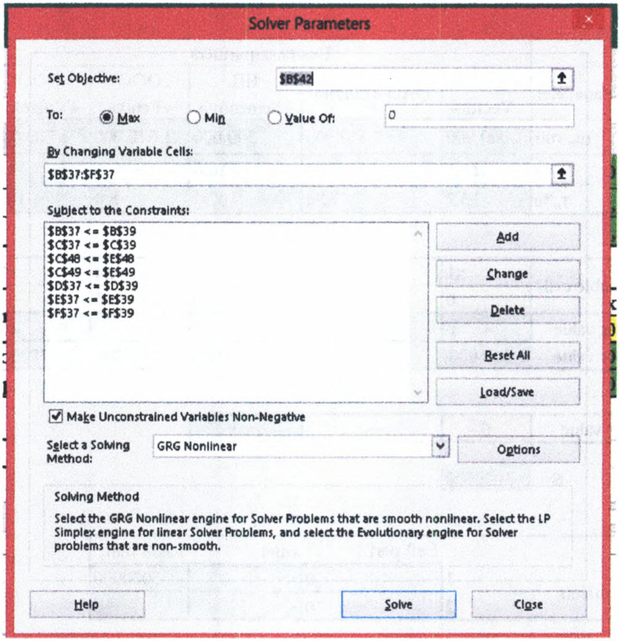


Fig. 11.13.

After launching "Solver" in the screen form, we will get the optimal solution of the problem.

35						
36	Changeable cells	x1	x2	x3	x4	x5
37		1000000	1450000	0	1 320 000	1230000
38	Lower value	1000000	1450000	0	1320000	1230000
39	Upper value	1 000 000	1 450 000	500 000	1 320 000	1 780 000
40						
41						
42	The OF value	679630	--->	max		
43						

Fig. 11.14.

Exercises for independent work

1. Three types of animal feed are used: A, B and C. Each animal requires at least 800 grams of fat, 700 grams of protein and 900 grams of carbohydrates per day. The content of 1 kilogram of each protein and carbohydrate (gram) feed is given in the table:

Content of 1 кg.	Mixed fodder		
	A	B	C
Fats	$100+10a$	200	300
Proteins	170	$100+10a$	110
Carbohydrates	380	400	$100+10a$
Cost of 1 kg	31	23	20

How many kilograms of each type of feed must be taken for each animal to have a minimum cost? Create a mathematical model for this LP problem and solve it by MS Excel. Here parameter a is the variant number (or an ordering number of a student in the list of the academic group)

2. The dairy produces milk, yogurt and sour cream, packaged in bottles. The production of 1 ton of milk, kefir and sour cream is required, respectively, $1000+a$, $1000+a$ and $9400+a$ kg of milk. At the same time, the working time for 1 ton of milk and kefir spills is 0.18 and 0.19 machine hours. Special machines are used for 1 ton of sour cream for 3.25 hours. A total of 136,000 kilograms of milk can be used to produce whole-milk products. At the same time, the working time for 1 ton of milk and kefir spills is 0.18 and 0.19 machine-hours. Special machines are used for 1 ton of sour cream for 3.25 hours. A total of 136,000 kilograms of milk can be used to produce whole-milk products. The main equipment can be occupied for 21.4 machine-hours and the packaging machines for 16.25 hours. The profit from the sale of 1 ton of milk, kefir and sour cream is equal to 30, 22+a and 136 rubles, respectively. The factory must produce at least 100 tons of milk per day, packed in bottles. There are no restrictions on other products.

It is necessary to determine what products and quantities should be produced daily by the plant in order to maximize the profit from its sales.

3. In the garment factory, the fabric can be cut in several ways to make the right parts of sewing products. Let the first version of cutting 100 м^2 of fabric produce 6 parts of the 1st kind, 8 parts of the 2nd type, 16 parts of the 3rd kind, and the amount of waste in this version is 3 м^2. In the second variant of cutting 100 м^2 of fabric, 4 parts of the 1st form, 10 parts of the 2nd form, 8 parts of the 3rd type, and the amount of waste in this version is 5м^2. In the 3rd version of cutting 100м^2 of fabric 9 parts of the 1st kind, 8 parts of the 2nd kind, 6 parts of the 3rd kind, and the amount of waste in this version is $2+a$ м^2. Knowing that parts of the 1st type should be produced $160+a$ pieces, parts of the 2nd type should be manufactured $110+a$ pieces, parts of the 3rd type should be manufactured $180+a$ pieces, it is necessary to cut the fabric so that the necessary number of parts of each type is obtained with minimal overall waste.

12. Transportation problems

Statement of the transportation problem

At m dispatch stations $A_1, A_2, \dots, A_m$ is concentrated respectively $a_1, a_2, \dots a_m$ units of homogeneous cargo. The cargo should be transported to n destinations $B_1, B_2, \dots, B_n$, and in each of the points it is necessary to wind respectively $b_1, b_2, \dots, b_n$ of cargo units. The cost of shipping a unit of cargo from A_i $(i = \overline{1,m})$ to B_j $(j = \overline{1,n})$ specified and equals c_{ij} $(i = \overline{1,m},$ $j = \overline{1,n})$. The total cargo stock at all dispatching stations is considered equal to the total cargo requirement at all destination stations:

(12.1)

It is necessary to draw up a transport plan in such a way that the total cost of all traffic will be minimal.

All data will be summarized in the table 12.1.

Table 12.1

The points of destination / The points of departure	B_1	B_2	...	B_j	...	B_n	Reserves
A_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1
...	...	...	...	...	...	...	...
A_i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}	a_i
...	...	...	...	...	...	...	...
A_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m
needs	b_1	b_2	...	b_j	...	b_n	$\sum b_j = \sum a_i$

Denote by x_{ij} the number of units of cargo to be sent from A_i to B_j. Then the quantity of cargo that is planned to be delivered to B_j from all points of departure is

$$x_{1j} + x_{2j} + \ldots + x_{mj} = \sum_{i=1}^{m} x_{ij}, \quad 1 \le j \le n$$

Since the cargo requirement of the destination B_j is b_j units, then the equality must be fulfilled

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad 1 \le j \le n \cdot$$

In its extended form, the latter equations are a system of n linear equations with m unknowns:

$$\begin{cases} x_{11} + x_{21} + \cdots + x_{m1} = b_1, \\ x_{12} + x_{22} + \cdots + x_{m2} = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_{1n} + x_{2n} + \cdots + x_{mn} = b_n. \end{cases}$$

On the other hand, the total quantity of cargo sent from A_i to all points B_j, is the amount

$$x_{i1} + x_{i2} + \cdots x_{in} = \sum_{j=1}^{n} x_{ij} = a_i, \quad 1 \le i \le m \cdot$$

Thus, similar to a system of equations above, we get a system of m linear equations with n unknowns:

$$\begin{cases} x_{11} + x_{12} + \cdots + x_{1n} = a_1, \\ x_{21} + x_{22} + \cdots + x_{2n} = a_2. \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_{m1} + x_{m2} + \cdots + x_{mn} = a_m. \end{cases}$$

The obtained systems of equations shall be now combined and written as a system of $n + m$ linear equations with nm unknown:

$$\begin{cases} \sum\limits_{i=1}^{m} x_{ij} = b_j, \quad 1 \le j \le n, \\ \sum\limits_{j=1}^{n} x_{ij} = a_i, \quad 1 \le i \le m. \end{cases}$$

If the values x_{ij} are combined into the table 12.2, then we will get a transport matrix in which the ratios are easily checked: the sum of the elements x_{ij}, located in the i–th line is equal to the stock a_i at A_i; the sum of the elements x_{ij} of column j equals the b_j point B_j.

Table 12.2

The points of destination / The points of departure	B_1	B_2	...	B_j	...	B_n	Reserves
A_1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1j} x_{1j}	...	c_{1n} x_{1n}	a_1
...	c_{21} x_{21}	c_{22} x_{22}	...	c_{2j} x_{2j}	...	c_{2n} x_{2n}	a_2
A_i	...	...	...	...	...	...	...
•••	c_{i1} x_{i1}	c_{i2} x_{i2}	...	c_{ij} x_{ij}	...	c_{in} x_{in}	a_i
A_m	...	...	...	...	...	...	...
needs	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mj} x_{mj}	...	c_{mn} x_{mn}	a_m

It follows from the terms of the task that the cost F of all traffic is equal to the sum

$$F = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{ij}x_{ij} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Since the cost F should be minimal, naturally we come to the following problem of linear programming: among all non-negative solutions of a system of equations, find one in which the objective function F reaches the lowest value.

Example 12.1. From three warehouses, it is required to deliver homogeneous goods to stores. Let there are 50 units of cargo in the warehouse A_1, 40 units in the warehouse A_2, and 20 units in the warehouse A_3. The specified goods need to be shipped to the 4th consumers: B_1, B_2, B_3, B_4, whose needs are respectively 35, 25, 30, 25 units of goods. The cost of transportation from a warehouse to consumers is presented in the table:

Table 12.3

	B_1	B_2	B_3	B_4
A_1	3	2	4	6
A_2	2	3	1	2
A_3	3	2	7	4

It is necessary to draw up a transportation plan that will ensure minimum transportation costs. You must create a screen form to enter the task conditions.

In the presented form, each variable and each task coefficient is matched with a MS Excel cell:

- ✓ Task variables $x_{11}, x_{12}, ..., x_{34}$ correspond to cells E17:H19;
- ✓ The capital OF coefficients correspond to cells E26:H28;
- ✓ Left parts of the constraint system correspond to cells I17:I19 and E20:H20;
- ✓ Right side of the constraint system corresponds to cells K17:K19 and E22:H22.

Then the data is entered into the screen form: transport tariffs, stocks, requirements.

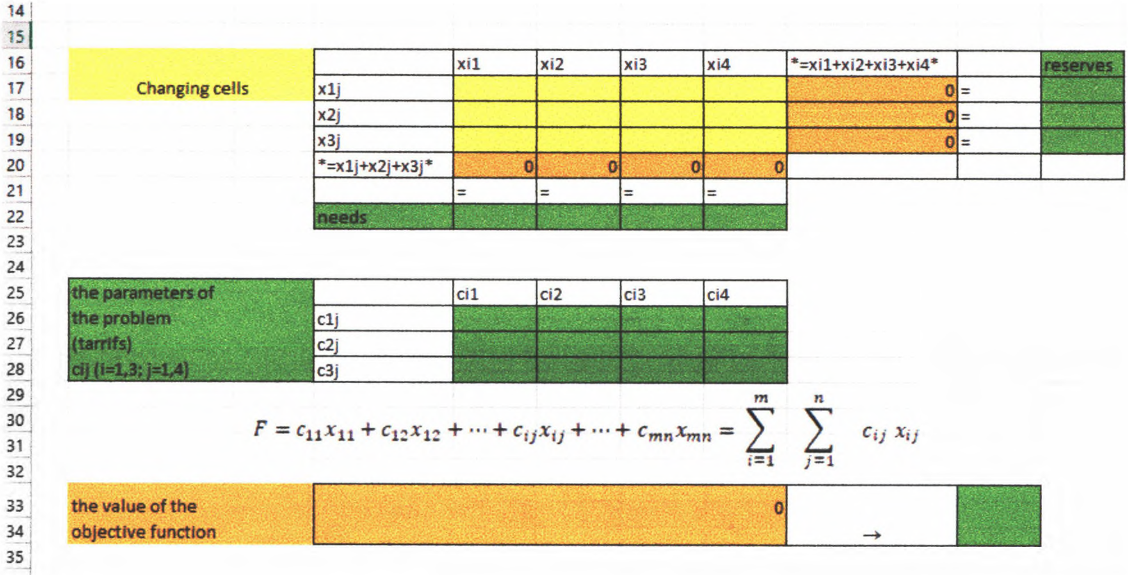
The left part of the system of constraints for the transportation problem

$$\sum_{i=1}^{3} x_{ij} = b_j,\ 1 \le j \le 4,$$

$$\sum_{j=1}^{4} x_{ij} = a_i,\ 1 \le i \le 3$$

are the sum of the cells for each row and column, assigned to the values of the task variables (E17:H19):

- ✓ I17 — constraint 1;
- ✓ I18 — constraint 2;
- ✓ I19 — constraint 3;
- ✓ E20– constraint 4;
- ✓ F20– constraint 5;
- ✓ G20– constraint 6;
- ✓ H20– constraint 7.



Changing cells
xi1
xi2
xi3
xi4
=xi1+xi2+xi3+xi4
reserves
x1j
x2j
x3j
=x1j+x2j+x3j
needs
the parameters of the problem (tarrifs) cij (i=1,3; j=1,4)
ci1
ci2
ci3
ci4
c1j
c2j
c3j
$F = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{ij}x_{ij} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$
the value of the objective function
→

Fig. 12.1.

15									
16			xi1	xi2	xi3	xi4	*=xi1+xi2+xi3+xi4*		reserves
17	Changing cells	x1j					0	=	500
18		x2j					0	=	400
19		x3j					0	=	200
20		*=x1j+x2j+x3j*	0	0	0	0			
21			=	=	=	=			
22		needs	350	250	300	200			
23									
24									
25	the parameters of		ci1	ci2	ci3	ci4			
26	the problem	c1j	3	2	4	6			
27	(tarrifs)	c2j	2	3	1	2			
28	cij (i=1,3; j=1,4)	c3j	3	2	7	4			
29									

$$F = c_{11}x_{11} + c_{12}x_{12} + \dots + c_{ij}x_{ij} + \dots + c_{mn}x_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

33	the value of the				
34	objective function	0	→	min	
35					

Fig. 12.2.

Using the notation of the corresponding cells in MS Excel, the formula for the constraint system can be written as the sum of the cells of each row and column, assigned to the task variables (E17:H19) as follows:

- ✓ Click the cell I17;
- ✓ Click the " f_x " and invoke "Insert function»;
- ✓ Choose in the field "Category" category "Math";
- ✓ In the "Select a function" select "SUM" and click "OK";
- ✓ In the appeared window "Function Arguments", in the line "Number 1" enter the expression E\$17:H\$17.

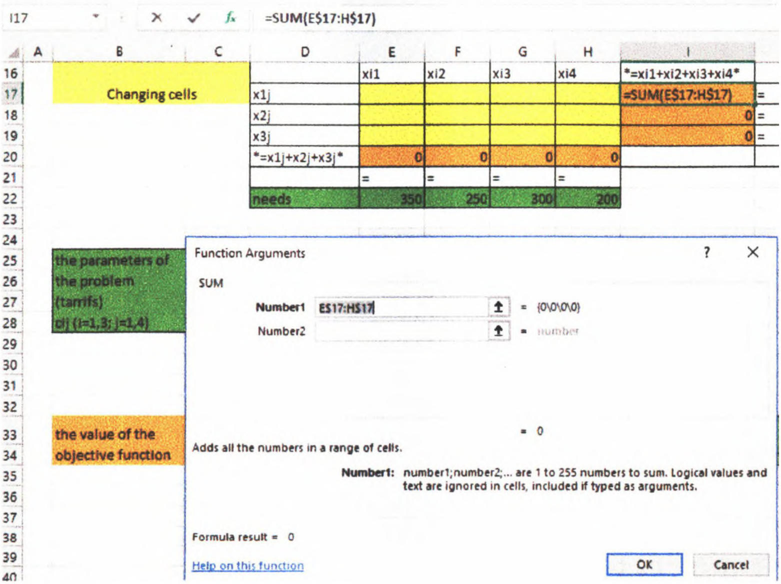


Fig. 12.3.

After entering the cells in the line "Number1" in the "SUM", a numeric value 0 will appear — the current value computed with the entered formula (see Fig.12.3).

Repeat the same steps for column amounts:

- ✓ Place the cursor in cell E20;
- ✓ Click the "f_x" and invoke "Insert function»;

✓ Choose in the field "Category" category "Math";
✓ In the "Select a function" select "SUM" and click "OK";
✓ In the appeared window "Function Arguments", in the line "Number 1" enter the expression E$17:E$19.

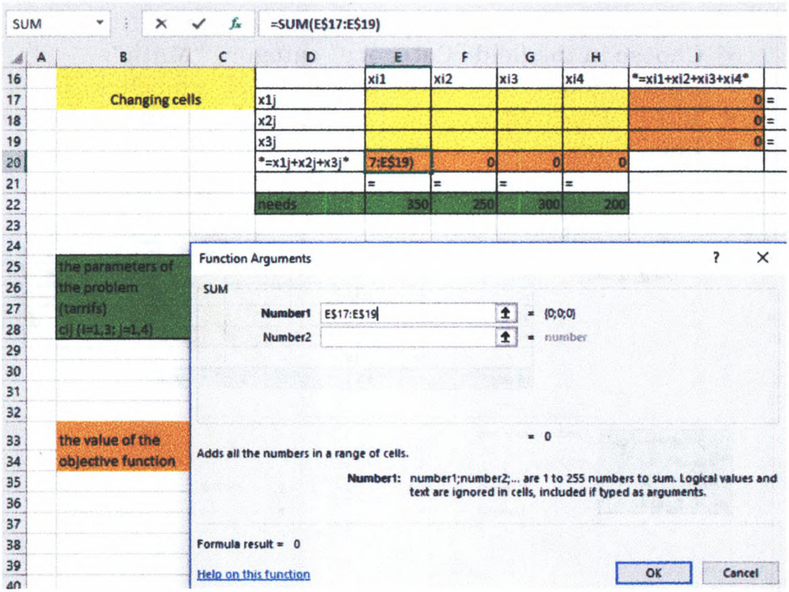


Fig. 12.4.

After entering the cells in the string "Number1" in the "SUM" will be a numeric value (see Fig.12.4), and in the screen form in the I20 cell the current value calculated by the entered formula, that is 0, will appear. For cells F20 – constraint 5; G20 – constraint 6; H20 – constraint 7; the formula entered in E20 can also be stretched by columns.

In cell D34, where the value of OF will be displayed, enter the formula

$$F = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{ij}x_{ij} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

for which this value will be calculated.

The formula for calculating the OF can be written as the sum of the products of each of the cells of the task variables (E17:H19), in the corresponding cells, the OF coefficients (E26:H28) are as follows:

- ✓ Click the cell D34;
- ✓ Click the "f_x" and invoke "Insert function ";
- ✓ Choose in the field "Category" category "Math";
- ✓ In the "Select a function" select "SUMPRODUCT" and click "OK";
- ✓ In the appeared window "SUMPRODUCT" in the line «Array 1» enter the expression E$17:H$19, and in the line «Array 2» — expression E26:H28.

After entering the cells in the string "Array 1" and "Array 2" in the window "SUMPRODUCT" will be a numeric value typed arrays (see Fig. 12.5), and in the screen form in cell D34 the current value calculated on the entered formula, 0 will appear.

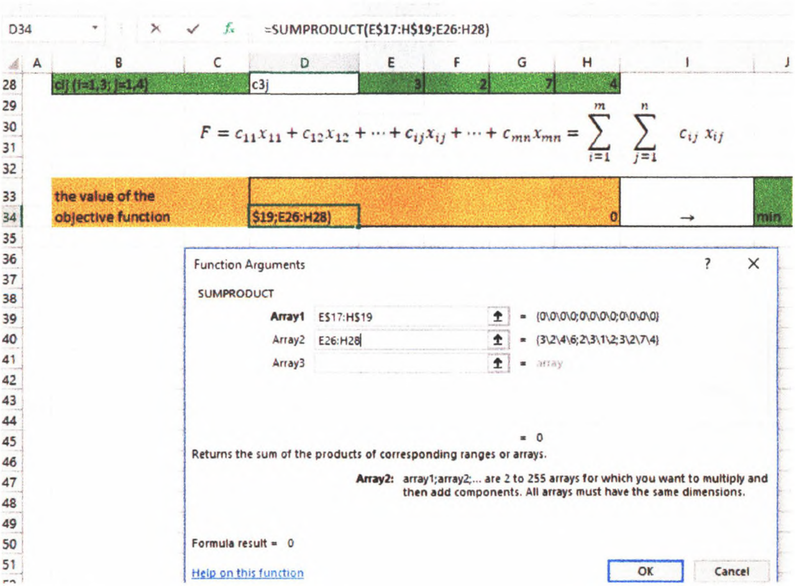


Fig. 12.5.

Formulas for the transportation problem are presented in the table 12.4.

Table 12.4

The object of the mathematical model	Expression in Excel
The variables of problem	E17:H19
The formula in the target cell E13	=SUMPRODUCT(E17:H19; E26:H28)
constraints on rows in cells I17 I18 I19	 =SUM(E17:H17) =SUM(E18:H18) =SUM(E19:H19)

For the implementation of further stages of solving the transport problem, it is necessary to call "Search for a solution" and take the following actions:

- ✓ Put the cursor in the field "Set Objective»;
- ✓ Enter the target cell address $D$34 or make one left click on the target cell in the on-screen form — this will be equivalent to entering the address from the keyboard;
- ✓ Enter the direction of optimizing the fit by clicking once the left mouse button on selector button the "MIN".

Enter $E$17:$H$19 in the "By Changing Variable Cells" field. The required addresses of the corresponding cells can be entered into the "Changing cells" field and automatically by selecting the corresponding cells of the variables directly in the screen form.

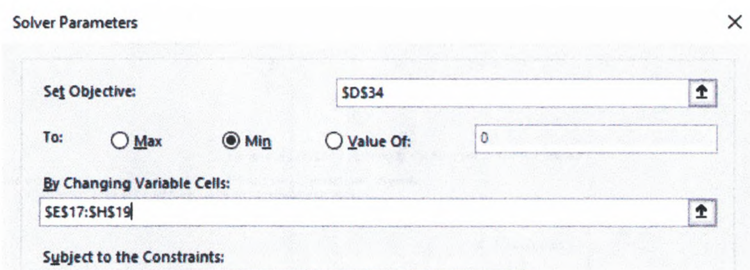


Fig. 12.6.

In the field "Subject to the Constraints", select the "Add" button, then the window "Add Constraint" will appear.

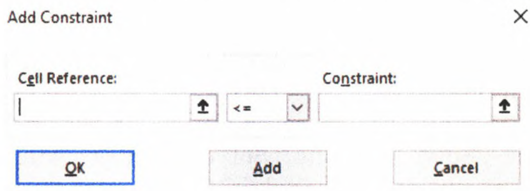


Fig. 12.7.

In the "Cell Reference" field, enter the cell address of the restriction system $I$17. This can be done both with the keyboard, and by the mouse highlighting all cells of variables directly in the screen form. In the sign box, open the list of suggested characters and select the sign "=". In the "Constraints" box, enter the cell address $K$17.

Fig. 12.8.

Similarly, you should enter all remaining string restrictions:

$I$18=$K$18
$I$19=$K$19

In the columns:

$E$20=$E$22
$F$20=$F$22
$G$20=$G$22
$H$20=$H$22

Then confirm all the above conditions by pressing the "OK" button.

The "Solver Parameters" window after you enter all the parameters of the problem is shown below (see fig. 12.9). If you want to modify or remove any constraints or boundary conditions, you can do so using the "Edit" or "Delete" buttons. In the condition of the task, there are restrictions $x \geq 0$ is necessary in the "Options" check "Make unconstraint variables non-negative". Choose “Simplex LP” in the window “Select a Solving Method”.

After entering the entire condition of the problem, you should click "Solve". In the "Solver results" window, select "Keep solver solution" and click "OK".

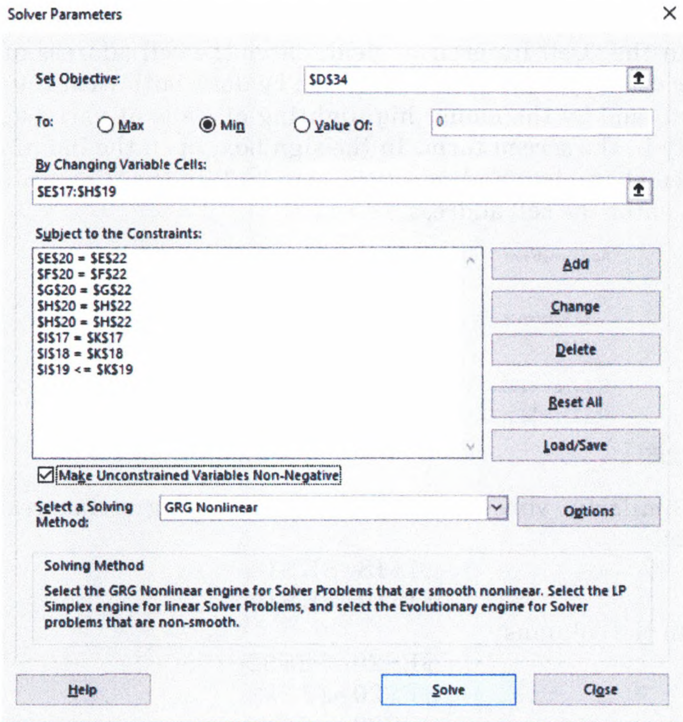
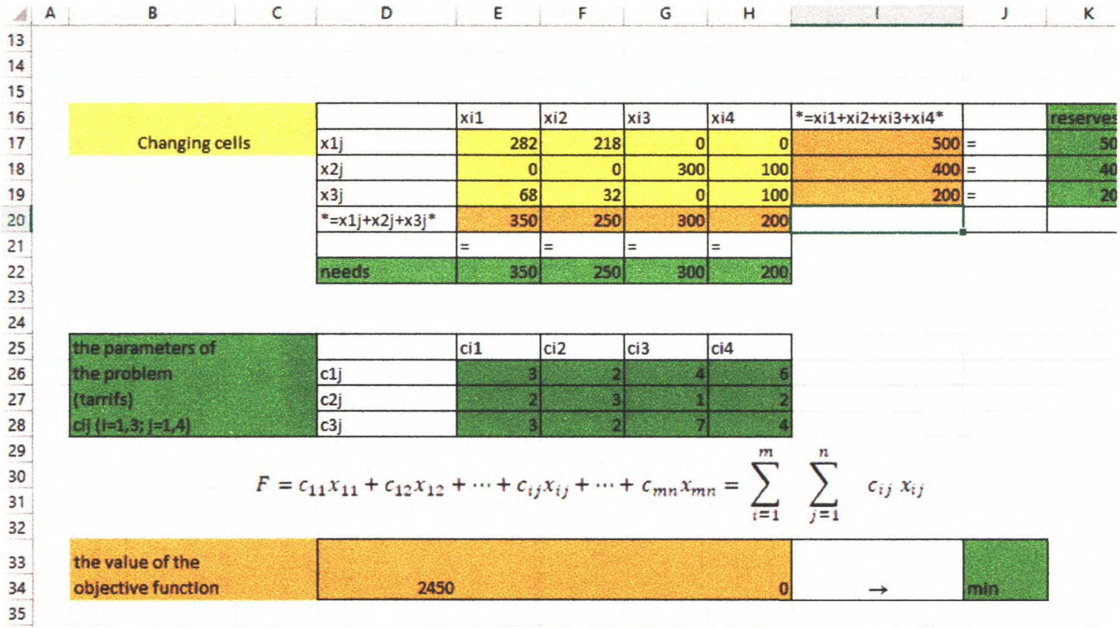


Fig. 12.9.

After all the described actions, we will get the optimal solution of the problem and the optimal value of the objective function (see fig. 12.10).



	B	D	E	F	G	H	I	J	K
16			xi1	xi2	xi3	xi4	*=xi1+xi2+xi3+xi4*		reserve
17	Changing cells	x1j	282	218	0	0	500	=	50
18		x2j	0	0	300	100	400	=	40
19		x3j	68	32	0	100	200	=	20
20		*=x1j+x2j+x3j*	350	250	300	200			
21			=	=	=	=			
22		needs	350	250	300	200			
25	the parameters of		ci1	ci2	ci3	ci4			
26	the problem	c1j	3	2	4	6			
27	(tarrifs)	c2j	2	3	1	2			
28	cij (i=1,3; j=1,4)	c3j	3	2	7	4			

$$F = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{ij}x_{ij} + \cdots + c_{mn}x_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

	B	D–H	I	J
33–34	the value of the objective function	2450 ... 0	→	min

Fig. 12.10.

Exercises for independent work

1. The company "Granit-holding" produces crushed stone and has 3 quarries. The reserves of gravel at quarries are equal to 800, 900 and 600 thousand m^3 respectively. Four construction companies, carrying out construction work on different objects of the same region, have ordered the supply of 300, 600, 650 and 500 thousand m^3. The cost of transportation of one thousand m^3 of rubble from each quarry for each object is given in the table:

Table 12.5

quarry	*Construction object*			
	1	2	3	4
1	8	4	1	7
2	3	a	7	3
3	$31-a$	5	11	8

It is necessary to draw up such a transport plan (the amount of gravel transported from each quarry to each construction site) so that the total cost of transportation is minimal.

The value of the unknown parameter a is equal to an option number (for example, an ordering number of a student in the list of academic group).

2. The metal shop needs to fulfill an urgent order for the production of parts. Each part is processed on four machines C1, C2, C3 and C4. Each machine can run any of the four workers P1, P2, P3, P4, however, each of them has a different scrap percentage on each machine. Records of the *Quality Control Department* show the percentage of defects per worker on each machine. It is necessary to divide the workers into machines so that the total percentage of the marriage (which is equal to the sum of the percentage of the marriage of all four workers) is minimal. What is this percentage?

The value of the unknown parameter a is equal to an option number.

Table 12.6

workers	machines			
	C1	C2	C3	C4
P1	2,3	1,9+***a***/20	2,2	2,7
P2	1,8+***a***/20	2,2	2,0	1,8+***a***/20
P3	2,5	2,0	2,2	3,0
P4	2,0	2,4	2,4–***a***/20	2,8

Identify as x_{ij}, i=1,2,3,4; j=1,2,3,4 variables that accept the values of 1 if the i-th works on the j-th machine. If this condition is not met, then $x_{ij} = 0$. The target function is:

$$2{,}3x_{11}+(1{,}9+\mathbf{a}/20)x_{12}+2{,}2x_{13}+2{,}7x_{14}+(1{,}8+\mathbf{a}/20)x_{21}+$$
$$+2{,}2x_{22}+2x_{23}+(1{,}8+\mathbf{a}/20)x_{24}+2{,}5x_{31}+2x_{32}+2{,}2x_{33}+3x_{34}+$$
$$+2x_{41}+2{,}4x_{42}+(2{,}4-\mathbf{a}/20)x_{43}+2{,}8x_{44}\to \min.$$

Introduce constraints. Each worker can work on one machine only, that is

$x_{11}+x_{12}+x_{13}+x_{14}=1;$
$x_{21}+x_{22}+x_{23}+x_{24}=1;$
$x_{31}+x_{32}+x_{33}+x_{34}=1;$
$x_{41}+x_{42}+x_{43}+x_{44}=1.$

Moreover, each machine has only one worker:

$x_{11}+x_{21}+x_{31}+x_{41}=1;$
$x_{12}+x_{22}+x_{32}+x_{42}=1;$
$x_{13}+x_{23}+x_{33}+x_{43}=1;$
$x_{14}+x_{24}+x_{34}+x_{44}=1.$

In addition, all variables must be integer and non-negative: $x_{ij} \geqslant 0$, x_{ij} — integers.

13. Financial functions: PV, FV, FVSHEDULE

13.1. Overview and fundamental concepts

Mathematical finance, also known as *quantitative finance*, is a field of applied mathematics concerned with financial markets. Classic mathematical finance (or *mathematics of credit*) deals with interest calculations, promissory notes, certificates of deposit (CD), bonds, analysis of cash flows, loans, and investments. A lot of such problems, say, basic investment problems, can be effectively solved by *Microsoft Excel (MS Excel) financial functions*. Such skills are in great demand by employers and very useful in a career of financier.

The *face* (or *nominal*) value of money may not always be what it seems. A key consideration is the *time value of money* (TVM). This concept involves calculating the value of money in the past, present, or future. It's based on the premise that money increases in value over time because of interest earned by the money. In other words, a dollar invested today will be worth more tomorrow. For example, imagine that your rich uncle decided to give away some money and asked you to choose one of the following options:

- Receive $8,000 today.
- Receive $9,500 in one year.
- Receive $12,000 in five years.
- Receive $150 per month for five years.

If your goal is to maximize the amount received, you need to consider not only the face value of the money but also the *time value of the money when it arrives in your hands.*

The TVM depends on your perspective. In other words, you're either a lender or a borrower. When you take out a loan to purchase an automobile, you're a borrower, and the institution that provides the funds to you is the lender. When you invest money in a bank savings account, you are a lender; you are lending your money to the bank, and the bank is borrowing it from you. Several concepts contribute to the TVM:

Present value (PV): This is the *principal* amount. If you deposit $5000 in a bank savings account, this amount represents the *principal or present value,* of the money you invested. If you borrow $15000 to purchase a car, this amount represents the principal or present value, of the loan. Present value may be positive or negative. PV is an amount today that is equivalent to a future payment, or series of payments, that has been *discounted* by an appropriate interest rate. Since money has *time value*, the present value of a promised future amount is worth less the longer you must wait to receive it. The difference between the two depends on the number of compounding periods involved and the interest (discount) rate.

Future value (FV): This is the *principal* plus *interest*. If you invest $5,000 for five years and earn 3% annual interest, your investment is worth $5,796.37 at the end of the five-year term. This amount is the future value of your $5,000 investment. If you take out a three-year car loan for $15,000 and make monthly payments based on a 5.25% annual interest rate, you pay a total of $16,244.97. This amount represents the principal $15,000 plus the interest $1,244.97 you paid. Future value may be positive or negative, depending on the perspective (lender or borrower). In other words, future value is the amount of money that an investment made today (the present value) will grow to by some future date. Since money has time value, we naturally expect the future value to be greater than the present value. The difference between the two depends on the number of compounding periods involved and the going interest rate. Thus, we can say, that the FV is related to *increment* — the process of the initial amount increasing because of the interest accumulation.

Payment (PMT): This is either principal or principal plus interest. If you deposit $100 per month into a savings account, $100 is the payment. If you have a monthly mortgage payment of $1,025, this amount is made up of principal and interest. If payments are made at the end of a certain period (month, quarter, year, etc.), they are called *postnumerando* or ordinary. If payments are made at the beginning of each period, they are called *prenumerando.*

Interest rate (RATE): Interest is a percentage of the principal, usually expressed on an annual basis. For example, you may earn 2,5% annual interest on a bank CD (certificate of deposit). Or your mortgage loan may have a 6,75% interest rate. *Simple interest* is generally used in short-term financial transactions the deadline for which is less than a year.

$$FV = PV(1 + i^*n)$$
$$PV = FV/(1 + i^*n)$$

Here i is an interest rate, and n is a number of periods (e.g. 3 years). Simple interest is calculated on the original principal only. Accumulated interest from prior periods is not used in calculations for the following periods. Simple interest is normally used for a single period of less than a year, such as 30 or 60 days. *Compound* interest is used in long-term financial transactions with a maturity of more than one year:

$$FV = PV(1 + i)^n$$
$$PV = FV/(1 + i)^n$$

Compound interest is calculated each period on the original principal and all interest accumulated during past periods. Although the interest may be stated as a yearly rate, the compounding periods can be yearly, semiannually, quarterly, or even continuously. You can think of compound interest as a series of back-to-back simple interest contracts. The interest earned in each period is added to the principal of the previous period to become the principal for the next period. The power of compounding can have an astonishing effect on the accumulation of wealth. Unless otherwise stated, we think of compound interest rate.

Period (PER): This represents the point in time when interest is paid or earned (for example, a bank CD that pays interest quarterly or an auto loan that requires monthly payments). TVM is based on the concept that a dollar that you have today is worth more than the promise or expectation that you will receive a dollar in the future. Money that you hold today is worth more because you can invest it and earn interest. After all, you should receive some compensation for foregoing spending. For instance, you can invest your dollar for one year at a 6% annual interest rate and accumulate $1.06 at the end of the year. You can say that the FV of the dollar is $1.06 given a i=6% interest rate and a one-year period (i.e. PER is one year here). It follows that the PV of the $1.06 you expect to receive in one year is only $1.

Term (NPER) — the total number of payment periods: This is the amount of time of interest. A 12-month bank CD has a term of one year. A 30-year mortgage loan has a term of 360 months. The variable *n* in TVM formulas represents the number of periods (*NPER*). It is intentionally not stated in years since each interval must correspond to a compounding period for a single amount or a payment period for an annuity. The interest rate and number of periods must both be *adjusted* to reflect the number of compounding periods per year before using them in TVM formulas. For example, if you borrow $1,000 for 2 years at 12% interest compounded quarterly, you must divide the interest rate by 4 to obtain rate of interest per period (i = 3%). You must multiply the number of years by 4 to obtain the total number of periods (n = 8).

You can determine the number of periods required for an initial investment to grow to a specified amount with the formula:

$$NPER = \ln(FV / PV) / \ln(1 + i), \text{ where:}$$

PV = present value, the amount you invested
FV = future value, the amount your investment will grow to
i = interest per period

So, we have briefly reviewed five important variables, related to the concept of TVM. We can calculate the fifth value if we are given any four of: Interest Rate, Number of Periods, Payments, Present Value, and Future Value. Carrying out almost any financial transaction generates a movement of cash. Such movement can be characterized by the appearance of individual one-time payments or set of time-distributed payments and receipts, i.e. is considered as a stream of payments or *cash flow*.

Cash flow — sequence of payments distributed over time. Any financial transaction requires two streams of payments: inflow — receipt (income) and outflow — payments (expenses). In the financial analysis these flows are typically replaced with a bi-directional flow of payments, where cash inflows are considered to be positive, and outflows — negative. The simplest (basic) cash flow consists of a single outflow and subsequent inflow or single inflow with subsequent outflow, separated by a certain period (e.g., year, quarter, month, etc.). Examples of financial transactions with such flows of payments are deposits, loans, transactions with certain types of securities, etc.

Payment flows on the periodicity are divided into *regular* and *irregular*. *Regular* flow of payments is a flow with one-direction payments (e.g., inflow), and with equal intervals between payments. *Irregular* flow of payments is a flow with positive payments (inflow) and negative payments (outflow). The intervals between payments in this case may not be equal.

The simplest example of a regular flow of payments is a financial rent. *Financial rent* or *annuity* is defined as a flow of payments, so that all of them are positive and come in with regular intervals.

Before dealing with such problems in MS Excel, it's helpful to consider some simple examples, which are easily solved arithmetically.

Example 13.1.1.

You borrow $10 000 for three years at 5% annual interest compounded annually:

interest year 1 =

$$p * i * n = 10\,000 * 0{,}05 * 1 = 500$$

interest year 2 =

$$(p_2 = p_1 + i_1) * i * n = (10\,000 + 500) * 0{,}05 * 1 = 525$$

interest year 3 =

$$(p_3 = p_2 + i_2) * i * n = (10\,500 + 525) * .05 * 1 = 551{,}25$$

Total interest earned over the three years =

$$500 + 525 + 551.25 = 1576{,}25$$

Compare this to 1,500 earned over the same number of years using simple interest.

Example 13.1.2.

You can afford to put \$10 000 in a savings account today that pays 6% interest compounded annually. How much will you have 5 years from now if you make no withdrawals?

Solution.

PV = 10 000

i = 0,06

n = 5

Therefore,

$$FV = 10\,000*(1 + 0{,}06)^5 = 10\,000*(1{,}3382255776) =$$
$$= 13382{,}26$$

Example 13.1.3.

Financial institution offers to pay 6% compounded semi-annually. How much will your \$10 000 grow to in five years at this rate?

Solution.

Interest is compounded twice per year, so you must divide the annual interest rate by two to obtain a rate per period of 3%. Since there are two compounding periods per year, you must multiply the number of years by two to obtain the total number of periods.

PV = 10 000

i = 0,06 / 2 = 0,03

n = 5 * 2 = 10

So,

$$FV = 10\,000*(1 + 0{,}03)^{10} = 10\,000*1{,}343916379 = 13439{,}16$$

Example 13.1.4.

You want to buy a house after 5 years from now for \$150 000. Assuming a 6% interest rate compounded annually, how much should you invest today to yield \$150 000 in 5 years?

Solution.

FV = 150 000

i =0,06

n = 5

$$PV = 150{,}000\,[1 / (1 + 0{,}06)^5] =$$
$$= 150\,000*(1 / 1{,}3382255776) = 112088{,}73$$

Example 13.1.5.

You find another financial institution that offers an interest rate of 6% compounded semiannually (i.e. once in a half a year). How much less can you deposit today to yield $150 000 in five years?

Solution.

Interest is compounded twice per year, so you must divide the annual interest rate by two to obtain a rate per period of 3%. Since there are two compounding periods per year, you must multiply the number of years by two to obtain the total number of periods.

FV = 150 000

i = 0,06 / 2 = 0,03

n = 5 * 2 = 10

$$PV = 150\,000*[\,1 / (1 + 0{,}03)*10] =$$
$$= 150\,000*(1 / 1{,}343916379) = 111614{,}09$$

Example 13.1.6.

You put $10,000 into a savings account at a 9,05% annual interest rate compounded annually. How long will it take to double your investment?

Solution.

$$\ln(20{,}000 / 10{,}000) / \ln(1{,}0905) = \ln(2) / \ln(1{,}0905) =$$
$$= 0{,}69314 / 0{,}08663 = 8 \text{ (years)}.$$

Let's move to MS Excel financial functions. Among the full range of financial functions, the following group used for investment analysis and estimation of credit and loan operations can be distinguished. (Table 13.1.1).

Table 13.1.1

Format	Purpose
FVSCHEDULE (principal, schedule)	*Calculates the future value of an initial principal (investment) after applying a series of compound interest rates.*
FV(rate, nper , pmt , *pv* , *type*)	*Returns the future value of an investment based on periodic, constant payments and a constant interest rate.*

Format	Purpose
IRR(values, *guess*)	*Returns the internal rate of return for a series of cash flows represented by the numbers in values. These cash flows do not have to be even, as they would be for an annuity (income-positive values, costs-negative values); the cash flows must occur at regular intervals, such as monthly or annually.*
NPER(rate, pmt, pv, fv, *type*)	*Returns the number of periods for an investment based on periodic, constant payments and a constant interest rate.*
MIRR(values, finance_rate, reinvest_rate)	*Returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.*
NOMINAL(effect_rate, npery)	*Returns the nominal annual interest rate, given the effective rate and the number of compounding periods per year.*
CUMPRINC(rate, nper, pv, start_period, end_period, type)	*Returns the cumulative principal paid on a loan between start period and end period.*
CUMIPMT(rate, nper, pv, start_period, end_period, type)	*Returns the cumulative interest paid on a loan between start period and end period.*
PPMT(rate, per, nper, pv, fv, *type*)	*Returns the payment on the principal for a given period for an investment based on periodic, constant payments and a constant interest rate.*
PMT(rate, nper, pv, fv, *type*)	*Calculates the periodic payment for a loan based on constant payments and a constant interest rate.*
ISPMT(rate, per, nper, pv)	*Calculates the interest paid during a specific period of an investment.*
IPMT(rate, per, nper, pv, fv, *type*)	*Returns the interest payment for a given period for an investment based on periodic, constant payments and a constant interest rate.*
PV(rate, nper, pmt, fv, *type*)	*Returns the present value of an investment; the present value is the total amount that a series of future payments is worth now.*

Format	Purpose
RATE(nper, pmt, pv, fv, *type, guess*)	*Returns the interest rate per period of an annuity, using iteration method.*
XIRR(values, dates, *guess*)	*Returns the internal rate of return for a schedule of cash flows that is not necessarily periodic.*
XNPV(rate, values, dates)	*Returns the net present value for a schedule of cash flows that is not necessarily periodic.*
NPV(rate, values)	*Calculates the net present value of an investment by using a discount rate and a series of future payments (negative values) and income (positive values).*
EFFECT (nominal_rate, npery)	*Returns the effective (factual) annual interest rate, given the nominal annual interest rate and the number of compounding periods per year.*

The detailed description of financial functions arguments can be found in the Table 13.1.2.

Table 13.1.2

Argument	Purpose
Dates (date1, ...,dateN)	*Schedule of payment dates that corresponds to the cash flow payments.*
Values (value1, ..., value N)	*Series of cash flows — costs and incomes (negative and positive values) that corresponds to a schedule of payments in dates.*
Npery	*Number of compounding periods per year*
End_period	*The last period in the calculation.*
Nper	*The total number of payment periods in an annuity (function NPER).*
Start_period	*The first period in the calculation.*
Nominal_rate	*The nominal interest rate (function NOMINAL).*
Principal (pv, investment)	*Present value of an investment.*
First_period	*Date of the end of the first period.*
Per	*Period for which you want to find the interest and must be in the range 1 to nper.*

Argument	Purpose
Schedule	*An array of interest rates to apply.*
Pmt	*Payment made each period; it cannot change over the life of the annuity (function PMT).*
Guess	*The guess for what the rate will be (assumed to be 0,1%)*
PV	*Returned present value of an investment, initial amount of a deposit (function PV)*
Rate	*Interest rate (function RATE).*
Reinvest_rate	*Interest rate you receive on the cash flows as you re-invest them.*
Finance_rate	*Interest rate you pay on the money used in the cash flows.*
Type	*The timing of the payment: 0–payment at the end of the period (default), 1–at the beginning of the period.*
Effect_rate	*Factual interest rate (function EFFECT)*

It should be noticed that before solving these problems we should answer the following questions. Who is the owner of money? For example, in case of accumulation — is it depositor or bank? In case of a loan — is it debtor or creditor? In case of calculating value of the future payments — is it a buyer (payment for consumed product) or a seller (getting payments for a sold product)? How cash is received? If it is received by the owner, then it is a positive value; if they outflow —then it is negative. After responding to the given questions, we can use MS Excel financial functions for carrying out the effective financial estimations and interpreting returned results in a correct way.

Remark. If you deal with a cash flow, and number of payments is not equal to the number of interest compoundings (per annum), then you should apply a general formula from Mathematical Finance, not MS Excel financial function.

13. 2. Future value and present value

Example 13.2.1.

Suppose we deposit $37000 at an interest rate of 11,5%. Required: determine the deposit amount 3 years later with semi-annual compounding.

Solution.

Since we need to determine the deposit amount using a constant interest rate, we shall use the FV (*rate, nper, pmt, pv, type*) function. Let us determine the arguments of this function. Due to semi-annual compounding, the *rate* argument is equal to 11,5/2 (%). The total number of periods of compounding is 3*2 (*nper* argument). If we look at this problem as a depositor, the *pv* argument (the initial deposit amount), which is equal to $37000, must be entered as a negative number (-37000), because for the depositor this is considered as a cash outflow. But, if we look at it as the bank, this argument (*pv*) must be a positive number, since it is cash inflow for the bank. The *pmt* argument is not used since the deposit is not renewed. The *type* argument is equal to 0, because in these operations interest is charged at the end of the period (given by default). Thus, by the end of the third year, we will have:

=FV(11,5%/2;3*2;;-37000)=$51746,86

to the depositor this is revenue,

=FV(11,5%/2;3*2;;37000)=$-51746,86

to the bank this is expense, because the bank has to return the funds to the depositor, see Fig.13.2.1.

In practice, depending on the terms, interest may be charged several times per year, e.g. monthly, quarterly, etc. If interest is compounded several times per year, it is necessary to determine the total number of compounding and the interest rate for the period. Table 13.2.1 contains data for the most frequent compounding methods.

	A	B	C	D	E	F
1	Task. Determine the future value of the deposit					
2						
3	Deposit	pv	-37000			
4	Periodical payment	pmt	0			
5	Annual interest rate		11,50%			
6	Compoundings per year		2			
7	Interest rate for the period	rate	5,75%			
8	Deposit period, years		3			
9	Number of periods	nper	6			
10	Payment type	type	0			
11	Future value of the deposit	fv	51 746,86 ₽		51 746,86 ₽	

Calculation using the FV function:
=FV(C7;C9;C4;C3;C10)

Analytical calculation, using the formula:
=-(C3*(1+C7)^C9+C4*(1+C7*C10)*((1+C7)^C9-1)/C7)

Fig. 13.2.1.

Table 13.2.1

Compounding method	Total number of compounding periods	Interest rate for the compounding period, %
Annually	m	i
Semiannually	m*2	i/2
Quarterly	m*4	i/4
Monthly	m*12	i/12
Daily	m*365	i/365

Example 13.2.2.

Determine the amount on a bank account if we annually add $20 thousand, for 5 years and the interest rate is 17%, compounded annually. Deposits are made at the beginning of each year.

Solution.

We must calculate the future value of fixed periodical payments, using a fixed interest rate, so we may use the FV function with the following arguments:

=FV(17%;5;-20000;;1)=$164 136,96

If the payments are made at the end of every year, the result would be:

=FV(17%;5;-20000)=$140 288

In the function under consideration the *pv* argument is not used because initially there was no money on the account.

Example 13.2.3.

Would a deposit of 85000 RUR be enough to buy a 160000 RUR car in 5 years? Interest is compounded quarterly; the annual interest rate is 12%. Investigate the problem and make calculations under different interest rates.

Solution.

Since we need to determine the future value of a deposit in 5 years, we will use the FV function.

=FV(12%/4;5*4;;-85000;0)=153519,45 RUR

As we can see, the value is not enough for the transaction. To make our dream come true we have two options: initially deposit a larger sum of money or use a bank which offers a higher interest rate. We shall not consider additional payments.

To determine the required sum let us present the initial conditions of the problem in tabular form and then use the *Goal Seek* option in the *Data tools* tab. An illustration is presented in Fig.13.2.2.

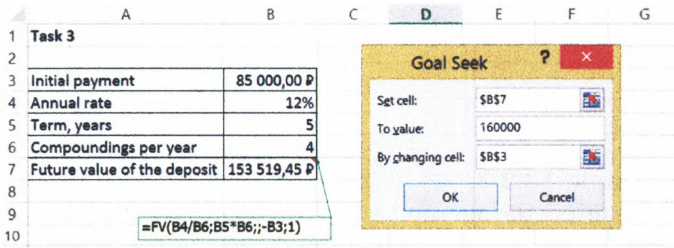


Fig. 13.2.2

After we confirm the entered data, 160000 RUR will appear in the cell B7, while our result appears in the cell B3, which is 88588,12 RUR. We may also use the *Goal Seek* option by changing the cell that contains the interest rate. But to analyse the way the interest rate influences the future value formula (which is dependent on it) we shall use another tool, the *Data Table* from the same *Data Tools* tab.

In addition to the initial conditions, which are presented in tabular form, let us determine the outline of the future data table; let us name the cells, enter the interest rates in the range D9:D16 and enter the future value formula into cell E8. Afterwards we shall start the data table tool and address the cells with the interest rates. The illustration of the MS Excel screen after the setting of data table parameters are presented in Fig.13.2.3.

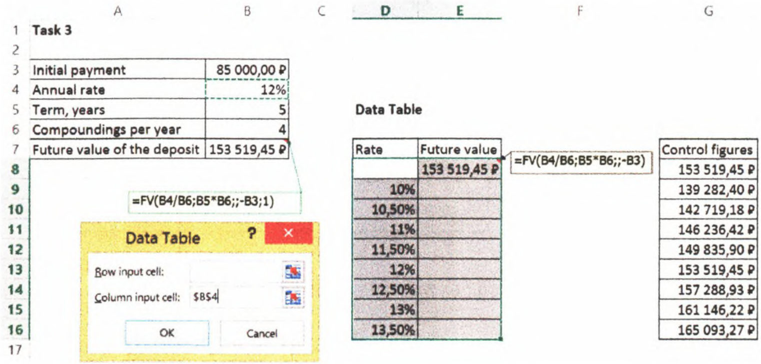


Fig. 13.2.3.

After confirming the parameters of the data table, the results (which comply with the control figures) appear in cells E9:E16. We may make a conclusion that annual interest rates less than 13% are not sufficient for the deposit to grow to the required amount of 160000 RUR. A rate of 13% provides 161146,22 RUR, while a rate of 13,5% provides 165093 RUR.

Example 13.2.4.

A bond with a par value of $50000 and 6-year term to maturity has the following order of percent charge: at the first year — 10%, at the next two years — 20%, at the remained three years — 25%. Determine the future cost of the bond considering a variable interest rate.

Solution.

Even though the interest rate changes over time it is still constant throughout each period of the same duration. For calculation of future value of an initial principal after applying a series of compound interest rates it is necessary to use the

FVSCHEDULE(principal; schedule)

The illustration of the solution of the problem is presented in Fig.13.2.4.

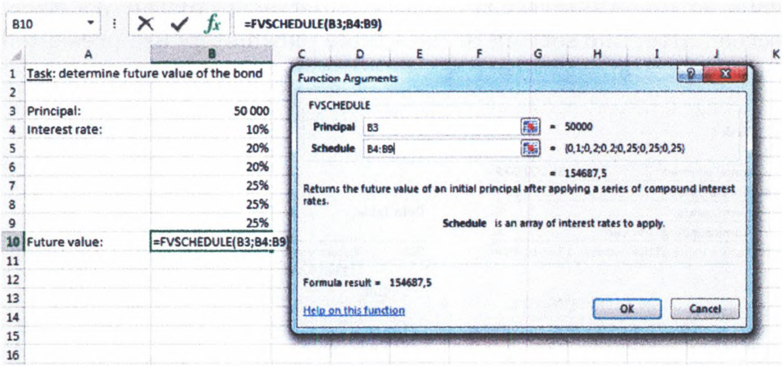


Fig. 13.2.4. FVSCHEDULE dialog box with data of the future cost of the bond

Answer to the task — \$154 687,50 — can be found with FVSCHEDULE function. The syntax for the *FVSCHEDULE* function should be entered in braces:

$$= FVSCHEDULE\,(50\,000;\,\{\,0{,}1;\,0{,}2;\,0{,}2;\,0{,}25;\,0{,}25;\,0{,}25\,\}) = \\ = 154687{,}50$$

The following formula is used for calculation of future value with the FVSCHEDULE function:

$$FVSCHEDULE = P * (1 + i_1) * (1 + i_2) * \ldots * (1 + i_n),$$

where: *FVSCHEDULE* — the future value of an initial principal after applying a series of compound interest rates;

P — principal;

n — number of periods;

i_k — interest rate at k period.

Calculation for the specified formula gives the same result:

$$FVSCHEDULE = 50000*(1 + 0{,}1)*(1 + 0{,}2)*(1 + 0{,}2)* \times \\ \times(1 + 0{,}25)*(1 + 0{,}25)*(1 + 0{,}25) = 154687{,}50.$$

Example 13.2.5.

Calculate the bond face value that is issued for 6 years with the same compounded interest rates provided in the previous example, if it is known that its future cost is \$154687,50.

Solution.

For the solution of the task we can use *Goal Seek* (from *What-If Analysis* group). Let basic data of a task be entered in accordance with fig. 4.4: in cells B4:B9 put interest rates; the cell B3 should storage the face value of the bond; in cell B10 enter the formula

=*FVSCHEDULE*(B3;B4:B9).

You should initiate procedure of goal seek and fill a dialog box in compliance with the data presented in fig. 13.2.5.

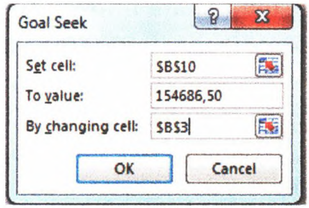


Fig 13.2.5. Determining the face value of the bond

After confirmation of input data in selection of parameter we will receive the face value of the bond — $50000 in a cell of B3.

Example 13.2.6.

The company requires $500000 in three years. Determine how much money it is necessary to deposit now to receive $500000 at the end of the third year. Interest rate is 12% per annum.

Solution.

To calculate the amount of the contribution we must define the data into a table. Insert PV function and put all the required information from the task (fig. 13.2.6). We will receive negative result since the required amount must be deposited.

While entering the data, the same value of deposit is obtained:

PV(12% ; 3; ; 500000) = -355 890,12

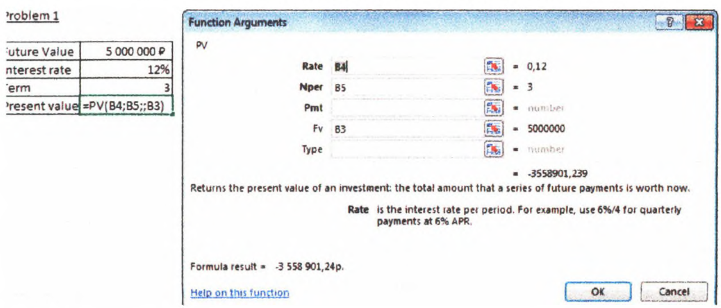


Fig. 13.2.6

Example 13.2.7.

The customer enters into a contract with a bank concerning 5 thousand dollars annuity payment at the end of each year within a five-year term. What sum must be deposited at the beginning of the first year by the client, to ensure that rent based on an annual interest rate of 20%?

Solution.

To calculate amount of the proposed investment based on the constant payments of 5 thousand rubles. Inserting the original data into PV function, we obtain:

= PV(20%; 5; 5000; 0; 0) = -14 953,06.

Recall that the "minus" sign shows us that money should be invested first.

Calculation via analytical formula brings us the same result:

$$PV = \frac{5000}{(1+0,2)} + \frac{5000}{(1+0,2)^2} +$$

$$+ \frac{5000}{(1+0,2)^3} + \frac{5000}{(1+0,2)^4} + \frac{5000}{(1+0,2)^5} = 14953,06.$$

Exercises for independent work

1. Find the value of the deposit if 8000 dollars invested in a bank for 29 months at 3% per annum and interest is compounded quarterly.
2. The annual interest rate is equal 3,4%. Initial capital is equal to $3000000. Find the value of the capital after three years if the interest compounded monthly.
3. At the beginning of every month the bank account is deposited by 11 thousand dollars. Determine the saved-up sum after 4 years at an interest rate of 4,1% per annum. Assume, that the interest is compounded monthly.
4. Bank suggests two options to invest money: on duration of 6 months with 14% interest rate per annum or on duration of 3 months with 13% per annum. Calculate which one will be more profitable for you if you plan to invest your money for 6 months (once for 6 months or twice for 3 months)?

5. The credit of 900 thousand dollars for the term of January 14th till April 14th of the year 2024 under 8,7% per annum is issued. Calculate the value of the payback.

6. Calculate future value of the bond of 200 thousand dollars issued for 4 years with the following order of percent charge: in the first year — 4,2%, in the next two years — 5,3%, in the last year — 5,7% per annum.

7. It is expected that the future cost of investment of 160 thousand dollars by the end of the fourth year will be 230 thousand dollars. Thus, for the first year the yield is 7%, for the second — 8%, for the fourth — 9%. Calculate the yield of investment for the third year, using the Goal Seek functionality of MS Excel.

8. Calculate the amount that needs to be put on deposit for 6 years to get 10 million dollars for the different scenarios of interest: monthly, quarterly, semi-annual and annual. The interest rate is 4,1% per annum.

9. An entrepreneur has received a bank loan at 6% per annum. What is the value of the loan, if the entrepreneur must transfer to the bank 253 000 dollars annually over 7 years?

10. To buy an apartment, a family plans in addition to their own savings of $12 000, take a bank mortgage loan for a period of 20 years at 11,5% per annum. The family can pay no more than $700 monthly on a loan. Assume, that the interest is compounded monthly.

- On what credit can the family count on? What can be the cost of the purchased apartment?
- What might be the cost of the purchased apartment, if you take a bank loan with other conditions: a) for 10 years at 10.5% per annum; b) for 15 years at 11% per annum?

14. Financial functions: NPV, NPER, RATE, NOMINAL, EFFECT

Example 14.1.

Let the investment in the project by the end of the first year of its implementation will be \$20000. In the next four years annual income is expected as follows: \$6000, \$8200, \$12600, \$18800. Calculate net present value of the project by the beginning of the first year if the interest rate is 10% per annum.

Solution.

The net present value of the project for periodic cash flows of a variable matter is calculated using *NPV* formula. Solution is presented in the Fig. 2.3.1. *NPV* of the project is calculated as follows:

= NPV(10%; -20000; 6000; 8200; 12600; 18800) =
= 13216,93.

This result represents earnings from the investment of 20 thousand rubles in the project if all costs are covered.

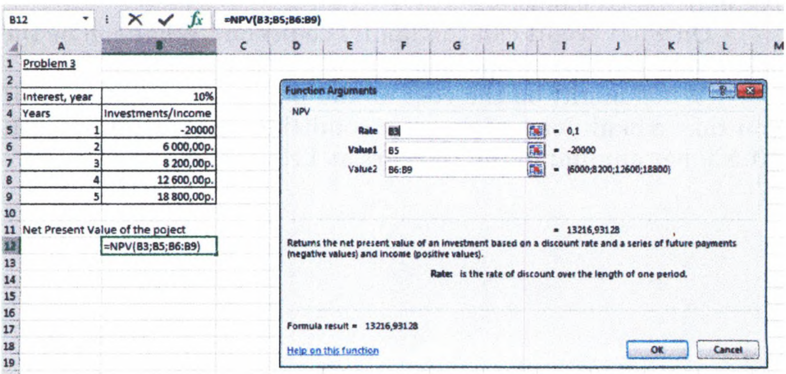


Fig. 14.1. NPV window

Analytical calculation of the problem gives the same result:

$$NPV = -\frac{20000}{(1+0{,}1)} + \frac{6000}{(1+0{,}1)^2} + \frac{8200}{(1+0{,}1)^3} +$$
$$+ \frac{12600}{(1+0{,}1)^4} + \frac{18800}{(1+0{,}1)^5} = 13216{,}93.$$

Example 14.2.

Investor considers 2 options of investing for 5 years:

• *Option 1* — initial investment 550000 RUR, expected earnings for 5 years are 100, 190, 270, 300, 350 thousand RUR consequently;

• *Option 2* — initial investment 650000 RUR, expected earnings for 5 years are 150, 230, 470, 180, 320 thousand RUR consequently.

Which of the options is preferable for the investor? Cost of capital is 15%.

Solution.

Assessment of the attractiveness of each option is executed by applying NPV function.

Since both projects involve an initial investment, subtract them from the result obtained using the NPV function (please note that you shouldn't discount the initial investment). To make the analysis of the solution of the problem easier, the original data can be represented in a table; in the appropriate cell enter the value of the formula with the function of NPV (see fig. 14.2). As the result of calculations, we find that the net present value of the investment in the second option is almost 22,000 RUR higher than in the first. Manual calculation gives the same results as the calculation using the NPV function.

1	Task 4		
2			
3		Option 1	Option 2
4	**Investment**	550,000 ₽	650,000 ₽
5			
6	*Revenue:*		
7	Year 1	100,000 ₽	150,000 ₽
8	Year 2	190,000 ₽	230,000 ₽
9	Year 3	270,000 ₽	470,000 ₽
10	Year 4	300,000 ₽	180,000 ₽
11	Year 5	350,000 ₽	320,000 ₽
12			
13	Interest	15%	15%
14			
15	Net Present Value	203.69103	225.39259
16			
17		=NPV(B12;B7:B11)-B4	
18			=NPV(C12;C7:C11)-C4
19			

Fig. 14.2

Option 1:

NPV (15%; 100000; 190000; 270000; 300000; 350000) – – 550000 = 203 691,03 RUR.

$$NPV = \frac{100000}{(1+0{,}15)} + \frac{190000}{(1+0{,}15)^2} + \frac{270000}{(1+0{,}15)^3} + \frac{300000}{(1+0{,}15)^4} + \frac{350000}{(1+0{,}15)^5} - 550000 = 203691{,}03\ RUR$$

Option 2:

NPV (15%; 150000; 230000; 470000; 180000; 320000) – – 650000 = 225 392,59 RUR.

$$NPV = \frac{150000}{(1+0{,}15)} + \frac{230000}{(1+0{,}15)^2} + \frac{470000}{(1+0{,}15)^3} + \frac{180000}{(1+0{,}15)^4} + \frac{320000}{(1+0{,}15)^5} - 650000 = 225392{,}59\ RUR.$$

Thus, the second option is more attractive for the investor.

To some extent, functions PV and NPV are similar. By comparing them, we can make the following conclusions:

1) Using function of PV, periodic payments assumed to be identical; in the NPV function they may be different;

2) Using function of PV bills and receipts occur both at the end and in the beginning of the period; NPV function assumes that all payouts are always equal and are made at the end of the period.

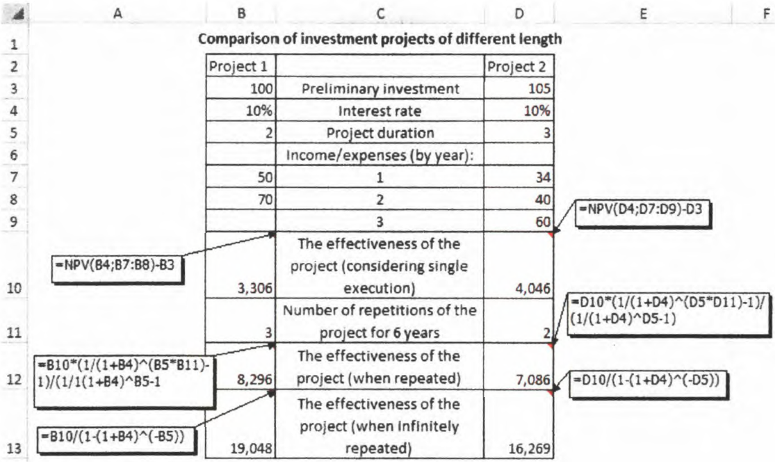
Example 14.3.

With two given options, choose one that is preferable for the investor. Cost of capital is 10%:

- *Option 1* (Project 1) — initial investment 100 million, expected earnings for *2* years are 50 and 70 million consequently;
- *Option 2* (Project 2) — initial investment 105 million, expected earnings for *3* years are 34, 40 and 60 million consequently.

<u>*Solution*</u>

To solve this problem, we pre-calculate the net present value of each option, using the NPV function and subtracting the initial investment. Then, considering the different duration of each option, we calculate the values of the efficiency of option by the corresponding formulas.



	A	B	C	D	E
1		Comparison of investment projects of different length			
2		Project 1		Project 2	
3		100	Preliminary investment	105	
4		10%	Interest rate	10%	
5		2	Project duration	3	
6			Income/expenses (by year):		
7		50	1	34	
8		70	2	40	
9			3	60	
10		3,306	The effectiveness of the project (considering single execution)	4,046	
11		3	Number of repetitions of the project for 6 years	2	
12		8,296	The effectiveness of the project (when repeated)	7,086	
13		19,048	The effectiveness of the project (when infinitely repeated)	16,269	

Fig. 14.3. Assessment of efficiency for different lengths

For a single option implementation, the preferred option is the second one. (NPV1 = 3,306; NPV2 = 4,046). But such a conclusion is premature (see Fig. 14.3). Calculation of the efficiency of each option for 6 years (even with their endless repetition) gives a result completely opposite: more attractive is the first option (first project):

$NPV_1\,(2,3) = 8{,}296$ $\qquad$ $NPV_1\,(2,\ ) = 19{,}048$

$NPV_2\,(3,2) = 7{,}086$ $\qquad$ $NPV_2\,(3,\infty) = 16{,}269$

Example 14.4.

Required: determine the net present value for the project for 5.04.2005, at a discount rate of 8%, if the cost of it for 5.08.2005, will amount to 90 million, and the expected income over the next months will be:

10 million for 10.01.2006;
20 million for 1.03.2006;
30 million for 15.04.2006;
40 million for 25.07.2006.

Solution.

Since in this case we deal with irregular variable expenses and income, to calculate the net present value of the project for 5.04.2005 you must use the XNPV function. The calculation of the net present value of irregular variable expenses and income using the XNPV function is performed by the formula:

$$XNPV = \sum_{i=1}^{n} \frac{P_i}{(1 + rate)^{\frac{d_i - d_1}{365}}} \qquad (14.4),$$

where: *XNPV* — the net present value for a schedule of cash flows that is not necessarily periodic;

Rate — rate of discounting;

d_1 — the initial date;

d_i — the ith payment date;

P_i — the ith payment;

n — the number of payments and receipts.

In order to find the solution we will first build a table with initial data. In the next column we will calculate the number of days from the initial date to the relevant payment. Then we will find the desired result using the XNPV function or by using the formula (14.4). We will get the value — 4 267 559,31. The illustration of the solution you can see in the fig.14.4.

Direct input of the parameters into XNPV function gives the same result:

=XNPV (8% ;{0;-90;10;20;30;40}; B4:B8) =
= 4,26755931 million.

The calculation by using the formula (14.4):

$$XNPV = \frac{-90000000}{(1+0{,}08)^{\frac{122}{365}}} + \frac{10000000}{(1+0{,}08)^{\frac{280}{365}}} + \frac{20000000}{(1+0{,}08)^{\frac{330}{365}}} +$$

$$+ \frac{30000000}{(1+0{,}08)^{\frac{375}{365}}} + \frac{40000000}{(1+0{,}08)^{\frac{476}{365}}} = 4267559{,}31 \; RUR.$$

Remarks.

1. Writing XNPV function explicitly, you cannot directly indicate an array of dates in any valid format as parameters. Be sure to refer to the cell where the date is given.

2. Analytical calculations according to the formulas should be performed in MS Excel worksheet (not by a calculator).

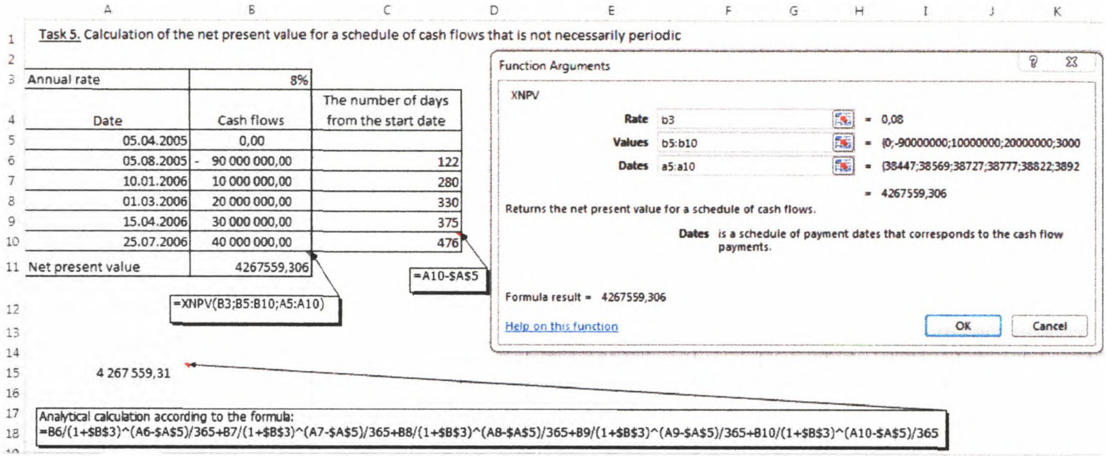


Fig. 14.4. Illustration of XNPV function implementation

Example 14.5.

Calculate the period when a deposit of \$100000 will be \$1000000, if the annual interest rate of the deposit is 13,5% and is compounded quarterly.

<u>*Solution.*</u> The quarterly interest rate is equal to 13% /4. For determining the general quantity of payment periods for one sum of a deposit we use the function NPER with the following arguments: rate = 13% /4; pv = -1; fv = 10. Zeros in current or future sums can be not counted; it is enough to save proportions between them. The meaning of function NPER is the number of periods, which is necessary for operations, in this case the number of quarters. To recognize the number of years we need to divide the result by 4:

=NPER(13% /4;;-1;10)/4=18

The illustration for the solution of the task is given in the Fig.14.5. It is easy to get the following formula for *NPER*:

$$NPER = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + rate)} \qquad (14.5)$$

Substituting values, we will see the match of results:

$$NPER = \frac{\ln\left(\frac{1000000}{100000}\right)}{\ln(1 + 0{,}0325)} = \frac{\ln 10}{\ln 1{,}0325} = \frac{2{,}302585}{0{,}031983} = 71{,}99393\,quaters$$

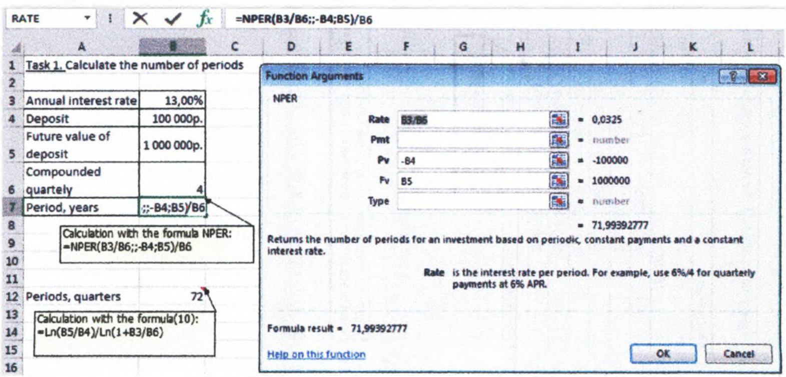


Fig. 14.5. NPER function and analytical formula for calculation of the number of periods

Example 14.6.

For covering its future expenditures, a firm establishes a fund. Resources are invested in the form of constant annual annuity. The value of each payment is $16000. Payments are discounted by the interest rate 11,2% annually. Find when the amount of fund will be $100000.

Solution.

To determine the general number of periods through which the necessary sum would be achieved, it is needed to use the function NPER with arguments: rate = 11,2%; pmt = -16; fv = 100. The result of the calculation is 5 years, in this time the amount of will achieve $100000:

= NPER(11,2%;-16;;100) = 5

The illustration of the solution of the task is given in the Fig.14.6

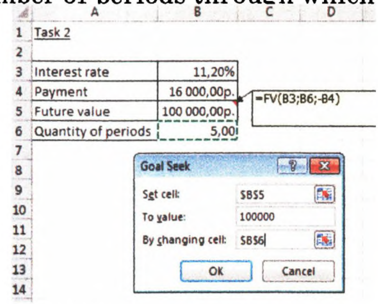


Fig. 14.6. FV function and the method of selecting data for getting the number of periods

Example 14.7.

Assume that to receive 1 million dollars in 2 years, a firm wants to deposit 250 thousand dollars right now and after that pay 25 thousand dollars monthly. Find the annual interest rate.

Solution.

In the given problem, the total amount is 1 million dollars, formed by the first payment to the future moment of time and following fixed monthly payments.

To determine the value of the monthly interest rate through the function RATE, we include the arguments: NPER = 2*12 = 24 (months); PMT = –25; PV = –250; FV = 1000. Then:

= RATE(24;-25;-250;1000) = 1,05%

To calculate the annual interest rate, given by the function RATE, it should be multiplied by 12: 1,05%*12 = 12,63%. The interest of the deposit should be not less than this value.

Illustration of application of function RATE in terms of formulas is given in the Fig.14.7. Pay attention that the function RATE is calculated by the method of successive approximation and may not to have a solution or may have several solutions.

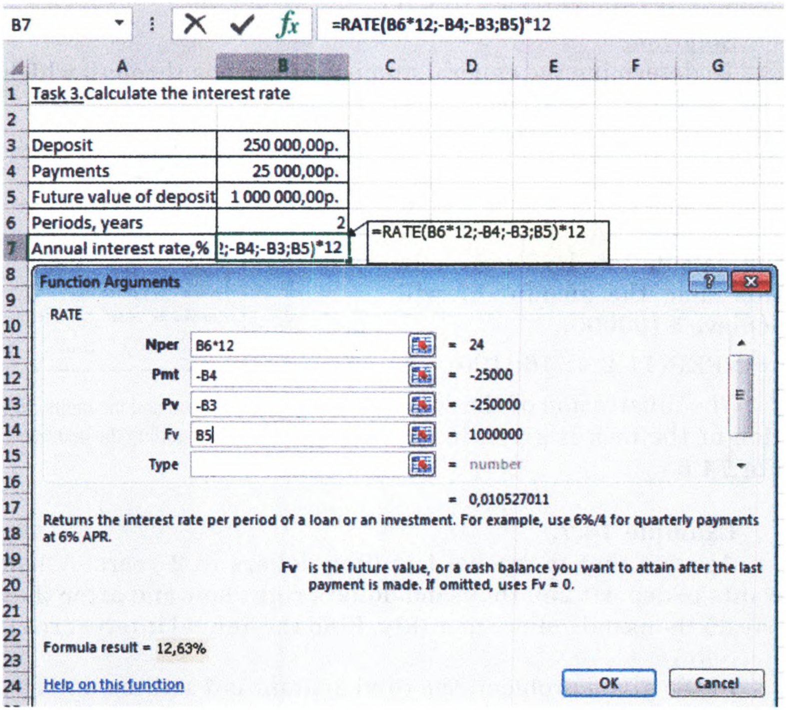


Fig. 14.7. RATE function

First, calculate the current volume of the investment with the rate, given by the argument of function RATE *guess* which is equal 10%. If the result is greater than 0, the value of the interest rate increases and the calculation of the current volume of investment repeats. If the result is less than 0, the next approximation of the value of the interest rate decreases.

The process concludes when the solution is within the accuracy of 0,0000001 or the quantity of interpretations exceeds 20. In the last case it is considered that there is no solution (#NUMBER! if an error) and for the second search of the solution we should change the value of the *guess* argument. It can work if we add a value from the interval between 0 and 1 in row of formulas or to shift a slider in the panel of function RATE, and in an appeared row to write a new value of argument *guess*.

Remarks.

1. We should remember that the results of functions NPER and RATE are numbers of periods and an interest rate of the current operations; that is why for annual results re-calculations are needed.

2. We should also remember getting an appropriate result if we work with functions NPER and RATE, arguments PV and FV, we must use contrary signs. The given requirement follows from the economic meaning of such operations.

Often in practice, it is necessary to compare conditions of financial transactions involving different periods of interest rate compounding. In this case, the adjusting of the respective interest rates to their annual equivalent is used. Real rate of return of the financial contract with interest compounded a few times per year is measured by the effective interest rate, which shows what relative income could be obtained during the year from the compounding of interest. Knowing the effective interest rate, it is possible to determine the amount of corresponding annual nominal interest rate.

For the calculation of these values the following functions are used:

NOMINAL(effect_rate, npery)
EFFECT(nominal_rate, npery).

Example 14.8.

Calculate an effective interest rate if the nominal interest rate equals 9% with interest compounded:

a) Semiannually;
b) Quarterly;
c) Monthly.

Solution.

For the calculation of an effective interest rate function EFFECT is to be used. Direct applying of arguments gives the following values:

a) =EFFECT(9% ; 2) = 9,2% ;

b) =EFFECT(9% ; 4) = 9,31% ;

c) =EFFECT(9% ; 12) = 9,38% .

NOMINAL is related to EFFECT as shown in the following equation:

$$EFFECT = \left(1 + \frac{Nominal\_rate}{Npery}\right)^{Npery} - 1$$

where Npery is the number of periods in a year when interest is compounded.

Applying this formula gives the same result. As an example calculate an effective interest rate for b).

$$EFFECT = \left(1 + \frac{0,09}{4}\right)^{4} - 1 = 1,0225^4 - 1 = 1,093083 - 1 = 9,31\%$$

Illustration of calculation is presented in the fig. 14.8.

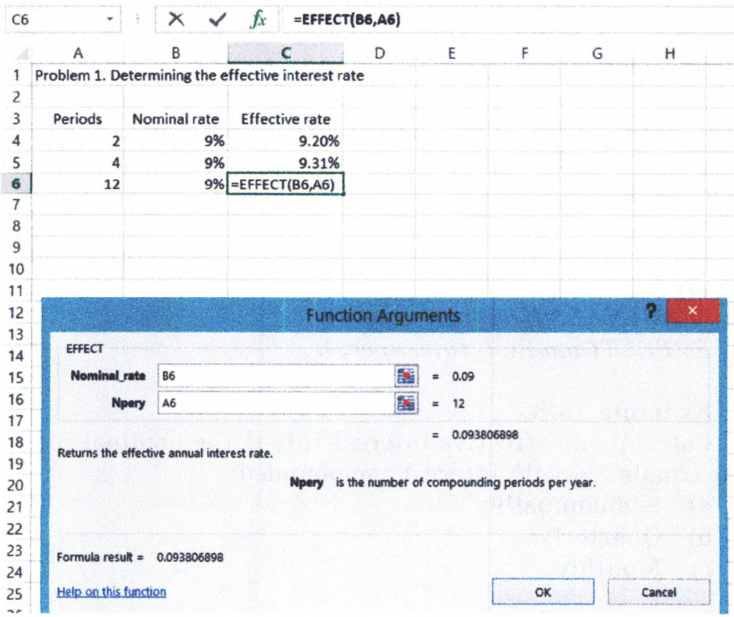


Fig. 14.8. EFFECT function

Common Errors.

1. #NUM! — Occurs if the nominal rate argument is $\leqslant 0$ or> 1, or if the npery argument is <1.
2. #VALUE! — Occurs if either of the supplied arguments is non-numeric.
3. #NAME? — Occurs when Analysis ToolPak add-in is not enabled in your MS Excel. You will need to enable the Analysis ToolPak if you wish to use the MS Excel Effect function.

Example 14.9.

Effective interest rate equals 16% and interest is compounded monthly. Calculate nominal interest rate.

Solution.

Nominal interest rate is calculated using the function NOMINAL:

=NOMINAL(16%;12) = 14,93%

Exercises for independent work

1. Determine the net present value of the project if the discount rate is equal to 7%. The project requires an initial investment in the amount of 3 million dollars. It is assumed that at the end of the first year the loss will amount to 600 thousand dollars, and in the next 3 years expected income will be as follows: 1100 thousand dollars, 2300 thousand dollars and 2800 thousand dollars, respectively.

2. Calculate the net present value of the project if:

- by the end of the first year its investments will amount to 31 thousand dollars, and the expected income in subsequent years will be: 4 thousand dollars, 13 thousand dollars and 22 thousand dollars, respectively; annual rate is 9%;
- solve the task with the same conditions, but given the preliminary investment in the project of 10 thousand dollars;
- analyze the resulting net present value of the project at various initial investment and different interest rates.

3. A loan is \$48000; interest is 12% per annum compounded monthly; monthly payments are \$2200. Calculate the period for paying off the loan.

4. Evaluate after how many years monthly payments which are equal 14000 dollars will bring the income of \$400000 with the interest rate 7,9% a year. Assume that the interest is compounded monthly.

5. There is a credit of 600 thousand dollars for 3,5 years. The interest rate is compounded semiannually. Determine the amount of interest rate for the period if we know that payback is 700 thousand dollars.

6. A client deposits \$10000 in the bank and at the end of a year he is planning to get back \$11000. Interest is compounded monthly. Define the interest rate of a deposit.

7. The balance of your account 4 years ago was \$23000. At the end of each year you added \$4100. Today the balance is \$65000. What was your interest rate?

8. Calculate an effective interest rate if the nominal rate equals 9% and interest is compounded:

a) 7000 times per year;

b) daily.

9. Effective rate equals 11%. Interest is compounded quarterly. Calculate nominal interest rate.

10. Interest rate for a bank deposit is 7% per year. Calculate an effective yield if the interest is compounded:

a) monthly;

b) quaterly;

c) annually.

Conclusion

Now, when we have a sufficient understanding of MS Excel in general, we can turn our attention to the true purpose in mathematical research of this splendid software environment — its use in data analysis.

It is possible to distinguish here several directions, where it is convenient to obtain both the results and their visual representation. These are theoretical probabilistic modeling of events and processes; descriptive statistics and statistical hypothesis testing; econometric methods, including the study of regression models; time series analysis, and more.

The spreadsheet processor MS Excel has a lot of capabilities. The question remains only in choosing a scientific field based on one's applied interest.

Recommended literature

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2. Frye C. *Microsoft Excel 2019 Step by Step.* — Microsoft Press, 2018
3. Winston W. *Microsoft Excel Data Analysis and Business Modeling (Office 2021 and Microsoft 365) (Business Skills)* — Microsoft Press, 2021

Учебное издание

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